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BOOK REVIEW

ON HEISENBERG'S SPECTRUM OF TURBULENCE

By

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(*Received January 15, 1951*)

1. Taylor (1938) first showed that the important properties of isotropic turbulence may be studied with the help of a spectrum function $F(k, t)$, representing the distribution of turbulent energy among the wave numbers corresponding to the fluctuations of velocity in turbulence. Since then one of the principal aims of the theory has been the determination of the spectrum function F. This so far not being possible from the equations of mechanics, empirical forms. suggested by various considerations are being tried for the present. Heisenberg (1949) has recently suggested such a form which satisfies the following general equation of decay

$$-\frac{\partial}{\partial t}\int_0^k F(k', t)\,dk' = 2(\nu+\eta_k)\int_0^k F(k', t)k'^2 dk', \tag{1}$$

here η_k stands for the turbulent viscosity associated with the transit of energy between the parts $(0 \to k,\ k \to \infty)$ of the spectrum. Heisenberg also puts

$$\eta_k = K\int_k^\infty \sqrt{\frac{F(k', t)}{k'^3}}\,dk' \tag{2}$$

K being a numerical constant. When the Reynolds number R is large, corresponding to the stage when the inertia terms in the equations of motion are the most important, ν, the kinematic viscosity may be neglected compared to η_k. A solution of the above equation having a homologous character has been suggested by Heisenberg in the form

$$F(k, t) \sim \frac{1}{\sqrt{t}}\, f(k\sqrt{t}) \tag{3}$$

An integral equation for the function f can then be easily obtained. Heisenberg further shows that $f(x) \sim x$ for small x, and $\sim x^{-5/3}$, for large x. This would make $F(k) \sim k$ for small k i.e. large eddies, and $\sim k^{-5/3}$ for theoretically an infinite value of Reynolds number $(\nu \sim 0)$. This latter result gives the well known Kolmogoroff spectrum; Lin (1947) and Batchelor (1949) have however found from different considerations that $F(k) \sim k^4$, for small k.

Heisenberg has also built up expressions which represent approximate solutions of (1) in good agreement with observational results. Recently Chandrasekhar (1949) has given rigorous solution of (1) in the form (3) with numerical tables for f.

In this paper a more general type of homologous solution than is represented by (3) will be discussed.

2. We assume a solution of the form

$$F(k,\ t) = \frac{1}{K^2}\cdot\frac{1}{k_0{}^3 t_0{}^2}\cdot\frac{s^3}{\tau^2} f(s,\ k/k_0) \tag{4}$$

where $s = s(\tau)$, $\tau = t/t_0$, and k_0, t_0 are constants. This is substituted in (1). Then introducing the Reynolds number R, and the variable x by the equations

$$\nu k_0{}^2 t_0 = 1/R, \quad k/k_0 . s = x \tag{5}$$

we obtain after some calculation

$$\int_0^x \left\{2 - 2\tau\,\frac{s_\tau}{s}\right\} f(x')dx' - \frac{\tau}{s}\cdot s_\tau . x f(x) = 2\left\{\frac{\tau}{s^2 R} + \int_x^\infty \sqrt{\frac{f(x')}{x'^{3/2}}}\,dx'\right\}\int_0^x x'^2 f(x')dx' \tag{6}$$

where s_τ denotes $ds/d\tau$. We now consider two cases.

Case 1. $R \gg 1$. We take the Reynolds number R to be very large corresponding to the stage of motion when the fairly high frequency eddies are predominant and the turbulent viscosity η_k is very large compared to ν, so that we may put $\nu = 0$, $R \to \infty$ in (6). The first summand in the bracket on the right of (6) may then be put equal to zero.

The resulting equation we can satisfy by putting (though not as necessary condition), firstly

$$\tau . \frac{s_\tau}{s} = c$$

which has as solution

$$s = \alpha\tau^c \tag{7}$$

α and c being constants; the function $f(x)$ then satisfies the equation

$$2(1-c)\int_0^x f(x')dx' - cxf(x) = 2\int_x^\infty \sqrt{\{f(x')/x'^3\}}dx' \int_0^x x'^2 f(x')dx'. \tag{8}$$

The solution (4) in the present case has the form

$$F(x,\ t) = \frac{1}{K^2}\cdot\frac{1}{k_0{}^3 t_0{}^2}(t_0/t)^{2-3c} f(k/k_0,\ (t/t_0)^c). \tag{9}$$

Here we have dropped a homology constant α independent of the time, and note that we have also the property first stated by Heisenberg that with $f(x)$, $\alpha^3 f(\alpha x)$ is also a solution. The particular case $c = 1/2$, in (9) gives the form of solution discussed by Heisenberg. We limit c to values not exceeding $2/3$.

Asymptotic values of $f(x)$:—The function $f(x)$ is determined by the above equation (9). We can find the behaviour of $f(x)$ as $x \to 0$ thus.

We note that as

$$f(x') \sim f(x) + (x'-x)f'(x) + \cdots \qquad (x \to 0),$$

$$\int_0^x x'^2 f(x')dx' \sim \frac{x^3}{3} f(x). \qquad (x \to 0).$$

Differentiating (8) with respect to x and using (10) we have

$$\frac{2-3c}{2x^2} - \frac{c}{2x}\frac{f'}{f} + \frac{1}{3}\frac{f^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \int_x^\infty \sqrt{\frac{f(x')}{x'^3}}\, dx' \qquad (x \to 0).$$

Differentiating this again with respect to x, multiplying by $x^3/^2$, and simplifying we obtain the equation

$$(2-3c)f^2 - \tfrac{1}{2}c\, xff' + \tfrac{1}{2}c\, x^2 ff'' - \tfrac{1}{2}c\, x^2 f'^2 + \tfrac{1}{6}x^{3/2}f^{5/2} - \tfrac{1}{6}x^{5/2}f^{3/2}f' - x^{3/2}f^{5/2} = 0.$$

We try the solution

$$f(x) \sim Ax^n \qquad (x \to 0)$$

Substituting in the above equation and equating the coefficient of x^{2n} to zero we obtain

$$n = (2-3c)/c$$

Hence we obtain the asymptotic behaviour

$$f(x) \sim \text{const. } x^{(2-3c)/c} \qquad (x \to 0). \tag{11}$$

As a check we note that the case $c = \frac{1}{2}$ gives the asymptotic form $f(x) \sim$ cost. x, of Heisenberg's solution. Further the case $c = \frac{2}{7}$ gives

$$f(x) \sim Ax^4 \qquad (x \to 0) \tag{12}$$

where A is independent of time, since f is supposed to contain the time only through x.

Substituting in (9) we now obtain

$$F(k,\, t) \sim A_0 k^4 \qquad (k \to 0) \tag{12a}$$

where A_0 is independent of t. This result was obtained by Lin and Batchelor. Lin and Batchelor have shown that A_0 is proportional to the Loitsiansky constant.

We now go back to equation (9) and find the behaviour of the analytic solution $f(x)$ as $x \to \infty$.

Putting
$$f(x') = e^{-w(x')}$$

and by following the method given by Heisenberg we find that the behaviour of f for large x is given by the behaviour for large x of solutions of

$$\frac{d}{dx}\left\{(2-3c)x^{3/2}f^{1/2} - cx^{5/2}\frac{f'}{f^{1/2}} - \frac{4x^3 f}{1-xf/f'}\right\} + 2x^3 f = 0 \qquad (x \to \infty)$$

If we try the solution

$$f(x) \sim \mathrm{A}x^{-n} \quad (n > 0,\ x \to \infty)$$

we find that the following equation to the highest order has to be satisfied

$$A\,\frac{6n-10}{n+1}\, x^{-(n-2)} - \tfrac{1}{2}A^{1/2}cx^{-(n-1)/2}.(n-3)(n-3+2/c) + \cdots = 0.$$

If the first term be the significant one, $(n-2)$ is less than $(n-1)/2$, so that n is less than 3. Then the asymptotic solution will correspond to

$$6n-10=0, \quad i.e. \quad n=5/3.$$

On the other hand the second term can be of higher order if $n>3$, but then the vanishing of the corresponding coefficient would require $n=3$, or $3-2/c$, both of which are then inadmissible. The assumption $n=3$ would lead to $A=0$. Hence the behaviour of $f(x)$ as $x \to \infty$ is given by $f(x) \sim Ax^{-5/3}$. This gives the Kolmogoroff spectrum. One gets indeed

$$F(k,\ t) \sim C.k^{-5/3} \tag{13}$$

C having time factor $t^{4c/3-2}$.

Thus we find that when the eddy viscosity is predominant which corresponds to the stage when the inertia terms in the equations of motion are the most important, the energy spectrum of turbulence proposed by Heisenberg admits of a multiplicity of homologous solutions of the form (9), characterized by a constant of homology c. These solutions have the property that for the low frequency part of the spectrum $F(k,\ t) \sim$ const. $k^{(2-3c)/c}(k \to 0)$, where the constant does not involve the time t. This for $c=2/7$, gives the fourth power law of Lin and Batchelor. But all these solutions ultimately lead to the same Kolmogroff spectrum $F \sim k^{-5/3}$, for $R \to \infty$. It appears permissible to suggest that the different low frequency spectra for different values of c, have their origin in the different modes of excitation of turbulence. They are, within the limits concerned, independent of the time but unstable. It may be useful from the above point of view to have numerical solutions of (8) for several values of c. Recently Chandrasekhar has given complete physical solutions for the case $c=1/2$.

Case 2. Let us now consider the case when η_k is negligible compared to ν. Then energy dissipation by friction becomes most important, and the contribution of inertia terms in the equations of motion unimportant.

We now neglect the second of the two terms on the right of (6). To satisfy the resulting equation we firstly put

$$s^2(2/\tau-2s_\tau/s)=c_1$$
$$s.s_\tau=c_2. \tag{14}$$

These two give consistent result when

$$c_2=\tfrac{1}{2}c_1=\tfrac{1}{2}\alpha, \quad s=\sqrt{(\alpha\tau)}=\alpha^{\frac{1}{2}}(t/t_0)^{\frac{1}{2}}. \tag{15}$$

This s is the same as in the *case 1* with $c=1/2$, which only means that Heisenberg's solution is valid at all the stages. Further we must have

$$\int_0^x f(x')dx'-\tfrac{1}{2}xf(x)=\frac{2}{\alpha R}\int_0^x x'^2 f(x')dx' \tag{16}$$

the solution of which is

$$f(x)=\text{const. } x \exp\{-2x^2/(\alpha R)\}. \tag{17}$$

Corresponding to this we have from (4), (15) and (17)

$$F(k,\ t)=\text{const. } \alpha^2\nu^2R^2k \exp(-2\nu k^2 t). \tag{18}$$

This form of solution was given by Heisenberg. Our present assumption does not yield anything new. We may note the following results.

The total energy contained in the part of the spectrum $(0, k)$, according to above formula is

$$E_k = \left(\frac{C}{t}\right)[1-\exp(-2\nu k^2 t] \tag{19}$$

where C is a constant, and C/t is the total turbulent energy in the spectrum. Batchelor's experiments, however, show that this part of the spectrum decays with time as $t^{-5/2}$.

We note that (14) are not necessary conditions. For instance, in this case we can strictly solve equation (1), *i.e.*

$$-\frac{\partial}{\partial t}\int_0^k F(k', t)dk' = 2\nu\int_0^k k'^2 F(k', t)dk'.$$

Transforming it into the form

$$\frac{\partial F(k, t)}{\partial t} = -2\nu k^2 F(k, t)$$

we find that the general solution of the above equation is

$$F(k, t) = F(k, t_0)\exp\{-2\nu k^2(t-t_0)\}. \tag{20}$$

8. Chandrasekhar has recently given a detailed discussion of the physical solutions of (1) and (2) in the form (8) after converting them into an ordinary differential equation in terms of new variables. We can, however, obtain a partial differential equation in terms of Chandrasekhar's variables if we donot assume the *ad hoc* form (8) of the solution. We use Chandrasekhar's variables

$$g = k^3 F(k, t); \qquad y = \int_0^k k'^4 F(k', t)dk' \tag{21}$$

both of which have the dimension $(\text{time})^{-2}$, as $F(k, t)$ has dimension $(\text{length})^3\,(\text{time})^{-2}$.

Substituting (2) in (1), differentiating partially with respect to k, and proceeding in the manner of Chandrasekhar we obtain

$$\frac{g}{k^2}\frac{\partial g}{t} = \frac{2ky}{k^2 g^{1/2}} - 2\left\{\nu + K\int_k^\infty g^{\frac{1}{2}}\frac{dk'}{k'^3}\right\}.$$

Again differentiating this equation partially with respect to k, simplifying, and writing simply t, for the product Kt, we obtain after some calculation finally the following partial differential equation for g

$$\frac{2}{g}\frac{\partial g}{t} - g\frac{\partial}{\partial y}\left(\frac{1}{g}\frac{\partial g}{\partial t}\right) = \frac{y}{g^{\frac{1}{2}}}(4+\partial g/\partial y) - 4g^{\frac{1}{2}} \tag{22}$$

This is the most general form of differential equation corresponding to the integral equation for the spectrum function given by Heisenberg.

We shall here briefly refer to a few types of solutions of this equation.

(i) First, we note that $g = Ay$ is an accurate solution of (22) if $A = 4/3$. This solution corresponds to Kolmogoroff spectrum as has been pointed out by Chandrasekhar.

(ii) If we try a solution of the type

$$g = p(y).q(t)$$

where p, and q are functional symbols, we find on substitution three equations which can be consistent only if $q = \text{const}$. This is the case of steady state spectrum already discussed by Chandrasekhar. (22) has no other solution in which y and t are separable.

(iii) Since t^2y is of no dimension, we expect from dimensional considerations similar solutions of the form $t^{-2} \times$ (function of t^2y).

We put therefore

$$y = t^{-2}f(\theta), \quad \theta = t^2 y. \tag{23}$$

Substitution of this in (22), and subsequent simplification gives the following ordinary differential equation for f

$$\theta f \frac{d}{d\theta}(f'/f) + f' - 2(\theta f'/f - 1) + \tfrac{1}{2}\theta(4/f^{1/2} + f'/f^{1/2}) - 2f^{1/2} = 0 \tag{24}$$

where $f' = df/d\theta$.

We will not attempt a discussion of this complicated equation but will investigate the relation between the solutions (3) and (23) of the equations (1) and (22). One may ask whether solution (23) necessarily implies a solution of the type (3) and vice versa.

Firstly, let

$$F(k, t) = (1/\sqrt{t})f(k\sqrt{t}) \tag{3}$$

then using (21), and (23) we obtain

$$y = \int_0^k k'^2 F(k', t)dk' = \int_0^{k\sqrt{t}} (1/t^2)(k\sqrt{t})^2 f(k\sqrt{t})d(k\sqrt{t}) = (1/t^2)Y(k\sqrt{t})$$

Inverting this equation we write

$$k\sqrt{t} = \eta(t^2y). \tag{24}$$

Then

$$g = \left(\frac{k^3}{\sqrt{t}}\right) f[\eta(t^2y)] = (1/t^2)f_2(t^2y)$$

by (24), which shows that (3) implies (23). Let us examine the converse.

Let

$$g = (1/t^2)f(t^2y)$$

Then from

$$y = \int_0^k (1/kt^2)f(t^2y)dk$$

we have considering t as a parameter

$$d(t^2y) = f(t^2y)dk/k$$

Integrating this equation we write

$$ks(t) = \phi(t^2y); \quad i.e. \quad k = \phi(t^2y)/s(t)$$

where $s(t)$ is an arbitrary function of t, and ϕ a functional symbol.

Then we have

$$F(k, t) = g/k^3 = \{f(t^2y)/t^2\}\{s^3(t)/\phi^3(t^2y)\} = \{s^3(t)/t^2\}\phi_2(k.s(t)). \tag{25}$$

Hence (23) generally implies the form (25) for $F(k, t)$. This suggests the form of the solution (4). We have seen that $s \propto \sqrt{t}$, for motion valid at all stages, but when ν is negligible a more general form of s is allowed.

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A SPECIAL METHOD FOR SOLVING THE EQUATION OF MESON IN THE FIELD OF PLANE ELECTROMAGNETIC RADIATION

By

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(Received February 7, 1951)

The object of this note is to obtain an exact solution of the equation of meson of spin one or zero, as formulated by Kemmer (1939), in the external field of a plane electromagnetic wave. The solutions of the corresponding problem for Dirac electron have been discussed by Volkow (1935), Sengupta (1947) and Taub (1949).

The Kemmer equation for meson of spin one or zero is

$$\left[\beta_\mu\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A^\mu\right)+\frac{mc}{\hbar}\right]\psi = 0 \tag{1}$$

where μ runs from 1 to 4, $x_4 = ict$, m and e are the mass and charge of the particle, $\hbar$ is Planck's constant divided by 2π, c is the velocity of light, A_μ $(A_4 = iA_0)$ is the four-vector potential describing the external electromagnetic field and β's are Hermitian matrices satisfying the following commutation rule

$$\beta_\mu\beta_\nu\beta_\rho+\beta_\rho\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\rho}+\beta_\rho\delta_{\nu\mu}. \tag{2}$$

The following identities, which can be readily deduced from the commutation rules (2), will be required for our subsequent discussion : If B^μ and D^μ are two four-vectors independent of β's

$$(\beta_\mu B^\mu)(\beta_\sigma D^\sigma)(\beta_\nu B^\nu) = (B_\mu D^\mu)(\beta_\nu B^\nu), \tag{3}$$

$$(\beta_\mu B^\mu)^3 = (B_\nu B^\nu)(\beta_\mu B^\mu), \tag{4}$$

$$(\beta_\mu B^\mu)^2(\beta_\nu D^\nu)+(\beta_\nu D^\nu)(\beta_\mu B^\mu)^2 = (B_\nu D^\nu)(\beta_\mu B^\mu)+(B_\mu B^\mu)(\beta_\nu D^\nu). \tag{5}$$

If A_μ describes a plane wave, we assume $A_\mu = A_\mu(\theta)$ = a function of the parameter θ only, where

$$\theta = \frac{1}{\hbar}k_\mu x^\mu, \tag{6}$$

$$k_\mu = k_x, k_y, k_z, ik_0/c \ (k_0 = \hbar\nu), \ k_\mu k^\mu = 0, \tag{7}$$

and from the transversibility condition we have

$$k_\mu A^\mu = 0. \tag{8}$$

Since A_μ is a function of the linear parameter θ only, it is suggested that we can assume the solution of the equation (1) in the form

$$\psi = U(\theta)\ \exp.\ \frac{i}{\hbar}S \tag{9}$$

where the matrix $U(\theta)$ is a function of the parameter θ only and the scalar function S is independent of β's. The substitution of (9) in (1) gives

$$\left[\beta_\mu\left(\frac{\partial S}{\partial x_\mu}-\frac{e}{c}A^\mu\right)-imc\right]U(\theta)=i\hbar\beta_\mu\frac{\partial U(\theta)}{\partial x_\mu}. \tag{10}$$

Let us now choose our S in such a manner that each side of this equation vanishes separately, thus

$$\left[\beta_\mu\left(\frac{\partial S}{\partial x_\mu}-\frac{e}{c}A^\mu\right)-imc\right]U(\theta)=0, \tag{11}$$

and so

$$\hbar\beta_\mu\frac{\partial U(\theta)}{\partial x_\mu}=(\beta_\mu k^\mu)\frac{\partial U(\theta)}{\partial \theta}=0. \tag{12}$$

From (12) it follows that

$$(\beta_\mu k^\mu)U=\text{constant}=K\text{ (say)} \tag{13}$$

where K is independent of the parameter θ.

The remainder of the calculation will be devoted to finding S and $U(\theta)$ by solving (11) subject to the condition (13). For this purpose we, at first, write

$$\frac{\partial S}{\partial x_\mu}-\frac{e}{c}A^\mu=R^\mu \tag{14}$$

and have from (11)

$$\beta_\mu R^\mu U(\theta)=imcU(\theta). \tag{15}$$

If we now multiply (15) by $\beta_\rho\beta_\nu R^\rho$ from the left, we have by (2)

$$\begin{aligned}imc\beta_\rho\beta_\nu R^\rho U(\theta)&=\beta_\rho\beta_\nu\beta_\mu R^\rho R^\mu U(\theta)=\tfrac{1}{2}\{\beta_\rho\delta_{\nu\mu}+\beta_\mu\delta_{\nu\rho}\}R^\rho R^\mu U(\theta)\\&=\beta_\mu R^\mu R_\nu U(\theta)=imcR_\nu U(\theta).\end{aligned}$$

Therefore

$$R_\mu U(\theta)=\beta_\nu\beta_\mu R^\nu U(\theta). \tag{16}$$

Operating again with R^μ from the left and summing over μ we get by (15)

$$R_\mu R^\mu U(\theta)=\beta_\nu\beta_\mu R^\mu R^\nu U(\theta)=-m^2c^2U(\theta),$$

or

$$\left[\left(\frac{\partial S}{\partial x_\mu}-\frac{e}{c}A^\mu\right)^2+m^2c^2\right]U(\theta)=0. \tag{17}$$

Thus the characteristic equation which determines S is

$$\left(\frac{\partial S}{\partial x_\mu}-\frac{e}{c}A^\mu\right)^2+m^2c^2=0. \tag{18}$$

To find S we define a function $f(\theta)$ by the equation

$$S=-p_\mu x^\mu-\hbar f(\theta), \tag{19}$$

where $p_\mu=p_x,\ p_y,\ p_z,\ iE/c$ and

$$p_\mu p^\mu=-m^2c^2. \tag{20}$$

Substituting (19) for S into (18) we obtain

$$\left(p_\mu+k_\mu f'(\theta)+\frac{e}{c}A_\mu\right)^2+m^2c^2=0, \tag{21}$$

where $f'(\theta) = \partial f / \partial \theta$. Since $k_\mu A^\mu = 0$, $k_\mu k^\mu = 0$ and $p_\mu p^\mu = -m^2c^2$, this equation readily gives

$$f'(\theta) = -\frac{e}{c}\frac{p_\mu A^\mu}{k_\sigma p^\sigma} - \frac{e^2}{2c^2}\cdot\frac{A_\mu A^\mu}{k_\sigma p^\sigma}. \tag{22}$$

But A_μ is a function of θ alone, therefore $f(\theta)$ can be obtained from (22) by simple quadrature.

It remains only to determine $U(\theta)$ for which we utilize (16) and have

$$\left(\frac{\partial S}{\partial x_\mu} - \frac{e}{c}A^\mu\right)U(\theta) = \beta_\nu\beta^\mu\left(\frac{\partial S}{\partial x_\nu} - \frac{e}{c}A^\nu\right)U(\theta)$$

and substituting (19) for S in this equation we obtain

$$\left(p^\mu + k^\mu f'(\theta) + \frac{e}{c}A^\mu\right)U(\theta) = \beta_\nu\left(p^\nu + k^\nu f'(\theta) + \frac{e}{c}A^\nu\right)\beta^\mu U(\theta).$$

We now multiply both sides of this equation by k_μ and sum over μ, and since $k_\mu k^\mu = 0$ and $k_\mu A^\mu = 0$, we have from (13)

$$(k_\mu p^\mu)U(\theta) = \left[\beta_\nu p^\nu + \beta_\nu k^\nu f'(\theta) + \frac{e}{c}\beta_\nu A^\nu\right]K = \left[\beta_\nu k^\nu f'(\theta) + \frac{e}{c}\beta_\nu A^\nu\right]K + K' \tag{23}$$

where K' is a constant independent of θ. Now operating (23) with $(\beta_\mu k^\mu)$ from the left we obtain

$$(k_\mu p^\mu)K = (\beta_\mu k^\mu)\left[(\beta_\nu k^\nu)f'(\theta) + \frac{e}{c}(\beta_\nu A^\nu)\right]K + (\beta_\mu k^\mu)K'. \tag{24}$$

Since by (3), (4) and (13)

$$(\beta_\mu k^\mu)(\beta_\nu k^\nu)K = (\beta_\mu k^\mu)^3 U(\theta) = (k_\mu k^\mu)(\beta_\nu k^\nu)U(\theta) = 0,$$

$$(\beta_\mu k^\mu)(\beta_\nu A^\nu)K = (\beta_\mu k^\mu)(\beta_\nu A^\nu)(\beta_\sigma k^\sigma)U(\theta) = (k_\mu A^\mu)(\beta_\nu k^\nu)U(\theta) = 0,$$

the equation (24) gives

$$K = \frac{(\beta_\mu k^\mu)}{(k_\sigma p^\sigma)}K'. \tag{25}$$

Hence from (23) we obtain

$$U(\theta) = \left[1 + \frac{(\beta_\nu k^\nu)(\beta_\mu k^\mu)}{(k_\sigma p^\sigma)}f'(\theta) + \frac{e}{c}\;\frac{(\beta_\nu A^\nu)(\beta_\mu k^\mu)}{(k_\sigma p^\sigma)}\right]u \tag{26}$$

where $u = K'/(k_\mu p^\mu)$ is the value of $U(\theta)$ in the absence of the field and satisfies the equation

$$(\beta_\mu p^\mu + imc)u = 0. \tag{27}$$

Thus the exact solution of the equation (1) in the present case is

$$\psi = \left[1 + \frac{(\beta_\mu k^\mu)^2}{(k_\sigma p^\sigma)}f'(\theta) + \frac{e}{c}\;\frac{(\beta_\nu A^\nu)(\beta_\mu k^\mu)}{(k_\sigma p^\sigma)}\right]u \exp.\ \frac{i}{\hbar}S$$

$$= \left[1 + \frac{(\beta_\mu k^\mu)^2}{(k_\sigma p^\sigma)}f'(\theta) + \frac{e}{c}\;\frac{(\beta_\nu A^\nu)(\beta_\mu k^\mu)}{(k_\sigma p^\sigma)}\right]\psi_0 \exp.\ [-if(\theta)] \tag{28}$$

where $f'(\theta)$ is given by (22), ψ_0 is the solution of (1) in the absence of the field and is expressed in the form

$$\psi_0 = u \exp. -\frac{i}{\hbar} p_\mu x^\mu. \tag{29}$$

The above solution (28) of the equation (1) may be interpreted as follows: If ψ_0 is the solution of (1) in the absence of any field, then the presence of the plane electromagnetic radiation field is performing a transformation of ψ_0, the transformation matrix being given by

$$\Lambda = \left[1+\frac{(\beta_\mu k^\mu)^2}{(k_\sigma p^\sigma)} f'(\theta) + \frac{e}{c}\,\frac{(\beta_\nu A^\nu)(\beta_\mu k^\mu)}{(k_\sigma p^\sigma)}\right] \exp.\,[-if(\theta)], \tag{30}$$

and consequently

$$\psi = \Lambda\psi_0. \tag{28$'$}$$

The exact solution of the equation complex-conjugate to (1) is then given by

$$\psi^* = \psi_0{}^*\Lambda^*. \tag{31}$$

With the above solution (28′) let us now discuss the behaviour of charge and current densities corresponding to a given value of p_μ. The four-current in the present theory is defined by

$$s_\mu = \psi^\dagger\beta_\mu\psi \tag{32}$$

where $\psi^\dagger = i\psi^*\eta_4$, $\eta_4 = 2\beta_4^2-1$ and

$$\eta_4\beta_k+\beta_k\eta_4 = 0\ (k = 1, 2, 3), \quad \eta_4\beta_4 = \beta_4\eta_4 = \beta_4. \tag{33}$$

Substituting the expressions for ψ and ψ^* from (28′) and (31) respectively, we obtain

$$s_\rho = i\psi_0{}^*\Lambda^*\eta_4\beta_\rho\Lambda\psi_0. \tag{34}$$

Now by (30) and (33) we have

$$\Lambda^*\eta_4\beta_\rho\Lambda = \left[1+\frac{(\beta_\mu k^{\mu *})^2}{(k_\sigma p^\sigma)} f'(\theta) + \frac{e}{c}\,\frac{(\beta_\mu k^{\mu *})(\beta_\nu A^{\nu *})}{(k_\sigma p^\sigma)}\right]\eta_4\beta_\rho\left[1+\frac{(\beta_\mu k^\mu)^2}{(k_\sigma p^\sigma)} f'(\theta) + \frac{e}{c}\,\frac{(\beta_\nu A^\nu)(\beta_\mu k^\mu)}{(k_\sigma p^\sigma)}\right]$$

$$= \eta_4\left[1+\frac{(\beta_\mu k^\mu)^2}{(k_\sigma p^\sigma)} f'(\theta) + \frac{e}{c}\,\frac{(\beta_\mu k^\mu)(\beta_\nu A^\nu)}{(k_\sigma p^\sigma)}\right]\beta_\rho\left[1+\frac{(\beta_\mu k^\mu)^2}{(k_\sigma p^\sigma)} f'(\theta) + \frac{e}{c}\,\frac{(\beta_\nu A^\nu)(\beta_\mu k^\mu)}{(k_\sigma p^\sigma)}\right]. \tag{35}$$

Again by (3), (4), (5) and $k_\mu k^\mu = 0$, $k_\mu A^\mu = 0$, we obtain

$$(\beta_\mu k^\mu)^2\beta_\rho+\beta_\rho(\beta_\mu k^\mu)^2 = k_\rho(\beta_\mu k^\mu)+(k_\mu k^\mu)\beta_\rho = k_\rho(\beta_\mu k^\mu),$$

$$(\beta_\nu k^\nu)(\beta_\mu A^\mu)\beta_\rho+\beta_\rho(\beta_\mu A^\mu)(\beta_\nu k^\nu) = (\beta_\mu k^\mu)A_\rho+(k_\mu A^\mu)\beta_\rho = A_\rho(\beta_\mu k^\mu),$$

$$(\beta_\mu k^\mu)^2\beta_\rho(\beta_\nu k^\nu)^2 = k_\rho(k_\mu k^\mu)(\beta_\nu k^\nu) = 0,$$

$$(\beta_\nu k^\nu)(\beta_\mu A^\mu)\beta_\rho(\beta_\sigma A^\sigma)(\beta_\lambda k^\lambda) = A_\rho(k_\sigma A^\sigma)(\beta_\lambda k^\lambda) = 0,$$

$$(\beta_\mu k^\mu)^2\beta_\rho(\beta_\sigma A^\sigma)(\beta_\nu k^\nu) = -(k_\sigma A^\sigma)\beta_\rho(\beta_\mu k^\mu)^2+k_\rho(k_\sigma A^\sigma)(\beta_\mu k^\mu)+(k_\mu k^\mu)\beta_\rho(\beta_\sigma A^\sigma)(\beta_\nu k^\nu) = 0,$$

$$(\beta_\nu k^\nu)(\beta_\sigma A^\sigma)\beta_\rho(\beta_\mu k^\mu)^2 = -(k_\sigma A^\sigma)(\beta_\mu k^\mu)^2\beta_\rho+k_\rho(k_\sigma A^\sigma)(\beta_\mu k^\mu)+(k_\mu k^\mu)(\beta_\nu k^\nu)(\beta_\sigma A^\sigma)\beta_\rho = 0.$$

Hence (34) assumes the form

$$s_\rho = \psi_0{}^\dagger\left[\beta_\rho+\frac{f'(\theta)}{(k_\sigma p^\sigma)}(\beta_\mu k^\mu)k_\rho+\frac{e}{c(k_\sigma p^\sigma)}(\beta_\mu k^\mu)A_\rho\right]\psi_0.$$

Substituting the expression for $f'(\theta)$ from (22), we obtain

$$s_\rho = \psi_0^\dagger \left[\beta_\rho + \frac{e}{c(k_\sigma p^\sigma)}(\beta_\mu k^\mu)A_\rho - \frac{e^2}{2c^2}\cdot\frac{(A_\mu A^\mu)}{(k_\sigma p^\sigma)^2}(\beta_\nu k^\nu)k_\rho - \frac{e}{c}\,\frac{(p_\mu A^\mu)}{(k_\sigma p^\sigma)^2}(\beta_\nu k^\nu)k_\rho\right]\psi_0$$

which can be rearranged in the form

$$s_\rho = \psi_0^\dagger \left[\beta_\rho + \frac{e}{c(k_\sigma p^\sigma)}\beta^\nu(k_\nu A_\rho - k_\rho A_\nu) - \frac{e^2}{2c^2(k_\sigma p^\sigma)^2}(A_\mu A^\mu)\beta^\nu k_\nu k_\rho \right.$$

$$\left. + \frac{e}{c(k_\sigma p^\sigma)^2}\beta^\nu k_\rho p^\lambda(k_\lambda A_\nu - k_\nu A_\lambda)\right]\psi_0. \qquad (36)$$

If we now write

$$F_{\mu\nu} = \frac{1}{\hbar}(k_\mu A_\nu - k_\nu A_\mu) \qquad (37)$$

and define

$$4\pi T_\mu{}^\nu = F_{\mu\lambda}F^{\lambda\nu} + \tfrac{1}{4}\delta_\mu{}^\nu F_{\lambda\sigma}F^{\lambda\sigma}$$

we have, in virtue of $k_\mu k^\mu = 0$ and $k_\mu A^\mu = 0$,

$$4\pi T_\mu{}^\nu = -\frac{1}{\hbar^2}(A_\lambda A^\lambda)k_\mu k^\nu. \qquad (38)$$

Hence s_ρ as given by (36) can be written in the form

$$s_\rho = \psi_0^\dagger\beta^\nu\left[\delta_{\nu\rho} + \alpha F_{\nu\rho} + 2\pi\alpha^2 T_{\nu\rho} + \frac{\alpha}{(k_\sigma p^\sigma)}k_\rho p^\lambda F_{\lambda\nu}\right]\psi_0 \qquad (39)$$

where $\alpha = e\hbar/c(k_\sigma p^\sigma)$.

Thus the effect of the presence of the electromagnetic radiation field on the expression for four-current may be considered as if it is performing a transformation of β-matrices, the transformation matrix being given by

$$\delta_{\nu\rho} + \alpha F_{\nu\rho} + 2\pi\alpha^2 T_{\nu\rho} + \frac{\alpha}{(k_\sigma p^\sigma)}k_\rho p^\lambda F_{\lambda\nu}. \qquad (40)$$

The last term is absent in the Dirac electron theory and this term is peculiar to the present theory of charged particle of spin one or zero.

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ON THE HANKEL TRANSFORMATION OF GENERALISED HYPERGEOMETRIC FUNCTIONS

By

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1. Hankel Transformations of various functions have been given from time to time. Recently Mr. Hari Shankar (1946) obtained many of these transformations from the generalised Hypergeometric function ${}_2F_2\ [\alpha,\beta;\gamma,\delta;-x]$. The object of the present note is to deduce Hari Shankar's result as a particular case of the transformation of the more general function ${}_3F_3\ [\alpha,\beta,\gamma;\delta,\lambda,\epsilon;-x]$. Needless to say that all the results deduced by Mr. Hari Shankar follow as particular cases from the results obtained by me.

2. Let

$$\phi(x) = \frac{x^{2\epsilon-\nu-3/2}}{\Gamma(\epsilon)2^{\epsilon-1}}\cdot {}_3F_3[\alpha,\beta,\gamma;\delta,\lambda,\epsilon;-\tfrac{1}{2}x^2], \tag{1}$$

$$\psi(x) = \frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)}\sum_{\alpha,\beta,\gamma}\frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)\Gamma(\alpha+\nu-\epsilon+1)}$$

$$\times\frac{x^{2\alpha+\nu-2\epsilon+\frac{1}{2}}}{2^{\alpha+\nu-\epsilon}}\cdot {}_3F_3[\alpha,\ \alpha-\delta+1,\ \alpha-\lambda+1;\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha+\nu-\epsilon+1;\ -\tfrac{1}{2}x^2]; \tag{2}$$

then $\phi(x)$ and $\psi(x)$ are Hankel Transformations of each other of order ν, provided that $\mathbf{R}(\epsilon)>0$, $\mathbf{R}(\nu+\alpha-\epsilon+1)>0$, $\mathbf{R}(\nu+\beta-\epsilon+1)>0$, $\mathbf{R}(\nu+\gamma-\epsilon+1)>0$ and $\mathbf{R}(\frac{3}{2}+\nu-2\epsilon+2(\alpha,\ \beta,\ \gamma))>0$, $\mathbf{R}(\epsilon)>\mathbf{R}(\frac{1}{2}\nu+\frac{1}{4})$, $\mathbf{R}(2\epsilon-\nu+\frac{3}{2}-2(\delta,\ \lambda))>0$.

Also we know that (Wright, 1935) as x tends to infinity,

$${}_pF_p\begin{bmatrix}\beta_1,\ \beta_2,\ \ldots\ \beta_p & ;\ \mp x\\ \alpha_1,\ \alpha_2,\ \ldots\ \alpha_p & \end{bmatrix}\sim x^{\theta}\exp(-x)+\sum_{n=1}^{n=p}a_n x^{-\beta_n},\ \text{where }\theta=\sum_{n=1}^{p}\beta_n-\sum_{n=1}^{p}\alpha_n.$$

3. We shall establish the theorem with the help of Operational Calculus. If

$$f(p) = p\int_0^\infty e^{-px}h(x)dx,\ \text{then } f(p)\doteqdot h(x).$$

Then (Tricomi, 1935) it is known that

$$\frac{1}{p^{\nu-1}}f\left(\frac{1}{p}\right)\doteqdot\int_0^\infty\left(\frac{x}{t}\right)^{\frac{1}{2}\nu}J_\nu\{2(xt)^{\frac{1}{2}}\}h(t)dt,$$

assuming that the integral converges. Let us take

$$h(x) = \frac{x^{\epsilon-1}}{\Gamma(\epsilon)}\cdot{}_3F_3[\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \epsilon;\ -x]$$

Then
$$f(p) = \frac{p}{\Gamma(\epsilon)} \int_0^\infty e^{-px} x^{\epsilon-1} \cdot {}_3F_3[\alpha, \beta, \gamma; \delta, \lambda, \epsilon; -x]\,dx$$
$$= \frac{1}{\Gamma(\epsilon)} \frac{1}{p^{\epsilon-1}} \int_0^\infty x^{\epsilon-1} e^{-x} \cdot {}_3F_3[\alpha, \beta, \gamma; \delta, \lambda, \epsilon; -x/p]\,dx$$
$$= \frac{1}{p^{\epsilon-1}} \cdot {}_3F_2(\alpha, \beta, \gamma; \delta, \lambda; -1/p),$$

term-by-term integration being permissible. Hence
$$\frac{1}{p^{\epsilon-1}} \cdot {}_3F_2[\alpha, \beta, \gamma; \delta, \lambda; -1/p] \doteqdot \frac{x^{\epsilon-1}}{\Gamma(\epsilon)} \cdot {}_3F_3[\alpha, \beta, \gamma; \delta, \lambda, \epsilon; -x]. \tag{4}$$

Making use of (3), we find that
$$\frac{\Gamma(\epsilon)}{p^{\nu-\epsilon}} \cdot {}_3F_2[\alpha, \beta, \gamma; \delta, \lambda; -p] \doteqdot \int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}\nu} J_\nu[2(xt)^{\frac{1}{2}}]\, t^{\epsilon-1} \cdot {}_3F_3[\alpha, \beta, \gamma; \delta, \lambda, \epsilon; -t]\,dt. \tag{4'}$$

Now it has been shown by Thomae, (1870) and Macrobert, (1939).
$$\sum_{\alpha,\beta,\gamma} \frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)} \Gamma(\alpha) z^{\alpha} \cdot {}_3F_2[\alpha, \alpha-\delta+1, \alpha-\lambda+1; \alpha-\beta+1, \alpha-\gamma+1; -z]$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\delta)\Gamma(\lambda)} \cdot {}_3F_2[\alpha, \beta, \gamma; \delta, \lambda; -1/z], \tag{5}$$

where $-\pi \leq amp\, z \leq \pi$ and a cross-cut is taken along the real axis from the origin to $-\infty$ to make the function one-valued, provides the analytic continuation of the left-hand side, valid for $|z| < 1$ into the region $|z| > 1$. The symbol $(=)$ shows that the expression on the two sides represent the same function, each in its own domain. Since $p > 1$,

$$\frac{1}{p^{\nu-\epsilon}} \cdot {}_3F_2[\alpha, \beta, \gamma; \delta, \lambda; -p]$$
$$= \frac{1}{p^{\nu-\epsilon}} \frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \sum_{\alpha,\beta,\gamma} \frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)} \frac{1}{p^{\alpha}} \cdot {}_3F_2\left[\begin{matrix}\alpha, \alpha-\delta+1, \alpha-\lambda+1 \\ \alpha-\beta+1, \alpha-\gamma+1\end{matrix}; -\frac{1}{p}\right]$$

and
$$\frac{1}{p^{\alpha+\nu-\epsilon}} \cdot {}_3F_2\left[\alpha, \alpha-\delta+1, \alpha-\lambda+1; \alpha-\beta+1, \alpha-\gamma+1; -\frac{1}{p}\right]$$
$$\doteqdot \frac{t^{\alpha+\nu-\epsilon}}{\Gamma(\alpha+\nu-\epsilon+1)} \cdot {}_3F_3[\alpha, \alpha-\delta+1, \alpha-\lambda+1; \alpha-\beta+1, \alpha-\gamma+1, \alpha+\nu-\epsilon+1; -t]$$

Hence
$$\frac{1}{p^{\nu-\epsilon}}\, {}_3F_2[\alpha, \beta, \gamma; \delta, \lambda; -p]$$
$$\doteqdot \frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \sum_{\alpha,\beta,\gamma} \frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha) t^{\alpha+\nu-\epsilon}}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)\Gamma(\alpha+\nu-\epsilon+1)} \cdot {}_3F_3\left[\begin{matrix}\alpha, \alpha-\delta+1, \alpha-\lambda+1; -t \\ \alpha-\beta+1, \alpha-\gamma+1, \alpha+\nu-\epsilon+1\end{matrix}\right]$$

Hence taking (4') into account, we get the result (Lerch, 1903)

$$\int_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}\nu} J_\nu[2(xt)^{\frac{1}{2}}]t^{\epsilon-1}\cdot{}_3F_3[\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \epsilon;\ -t]dt$$

$$=\frac{\Gamma(\delta)\Gamma(\lambda)\Gamma(\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\sum_{\alpha,\beta,\gamma}\frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha)x^{\alpha+\nu-\epsilon}}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)\Gamma(\alpha+\nu-\epsilon+1)}\cdot{}_3F_3\begin{bmatrix}\alpha,\ \alpha-\delta+1,\ \alpha-\lambda+1;\ -x\\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha+\nu-\epsilon+1\end{bmatrix} \quad (6')$$

Writing $\frac{1}{2}x^2$ for x and $\frac{1}{2}t^2$ for t, we get after slight modifications,

$$\frac{1}{\Gamma(\epsilon)2^{\epsilon-1}}\int_0^\infty J_\nu(xt)(xt)^{\frac{1}{2}}t^{2\epsilon-\nu-\frac{3}{2}}\cdot{}_3F_3[\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \epsilon;\ -\tfrac{1}{2}t^2]dt$$

$$=\frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\sum_{\alpha,\beta,\gamma}\frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)\Gamma(\alpha+\nu-\epsilon+1)}\ \frac{x^{2\alpha+\nu-2\epsilon+\frac{1}{2}}}{2^{\alpha+\nu-\epsilon}}$$

$$\times{}_3F_3[\alpha,\ \alpha-\delta+1\ \ \alpha-\lambda+1;\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha+\nu-\epsilon+1;\ -\tfrac{1}{2}x^2]; \quad (7)$$

in other words the result

$$\int_0^\infty J_\nu(xt)(xt)^{\frac{1}{2}}\phi(t)dt = \psi(x)$$

follows.

4. Let us next consider the integral

$$\frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_0^\infty J_\nu(xt)(xt)^{\frac{1}{2}}\sum_{\alpha,\beta,\gamma}\frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha)t^{2\alpha+\nu-2\epsilon+\frac{1}{2}}}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)\Gamma(\alpha+\nu-\epsilon+1)2^{\alpha+\nu-\epsilon}}$$

$$\times{}_3F_3[\alpha,\ \alpha-\delta+1,\ \alpha-\lambda+1;\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha+\nu-\epsilon+1;\ -\tfrac{1}{2}t^2]dt.$$

Writing $(2x)^{\frac{1}{2}}$ for x and $(2t)^{\frac{1}{2}}$ for t, this reduces to

$$\frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\ \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}\nu-\frac{1}{2}}}\int_0^\infty J_\nu[2(xt)^{\frac{1}{2}}]t^{\alpha+\frac{1}{2}\nu-\epsilon}\sum_{\alpha,\beta,\gamma}\frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)\Gamma(\alpha+\nu-\epsilon+1)}$$

$$\times{}_3F_3[\alpha,\ \alpha-\delta+1,\ \alpha-\lambda+1;\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha+\nu-\epsilon+1;\ -t]dt.$$

In (4′) let us put $\beta=\alpha-\delta+1$, $\gamma=\alpha-\lambda+1$, $\delta=\alpha-\beta+1$, $\lambda=\alpha-\gamma+1$ and write $\alpha+\nu-\epsilon+1$ for ϵ. We find that

$$\frac{1}{p^{\epsilon-\alpha-1}}\cdot{}_3F_2[\alpha,\ \alpha-\delta+1,\ \alpha-\lambda+1;\ \alpha-\beta+1,\ \alpha-\gamma+1;\ -p]$$

$$\doteqdot\int_0^\infty\frac{1}{\Gamma(\alpha+\nu-\epsilon+1)}J_\nu[2(xt)^{\frac{1}{2}}]x^{\frac{1}{2}\nu}t^{\alpha+\frac{1}{2}\nu-\epsilon}\cdot{}_3F_3\begin{bmatrix}\alpha,\ \alpha-\delta+1,\ \alpha-\lambda+1;\ -t\\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha+\nu-\epsilon+1\end{bmatrix}dt. \quad (8)$$

Therefore

$$\frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)}\int_0^\infty x^{\frac{1}{2}\nu}J_\nu[2(xt)^{\frac{1}{2}}].\sum_{\alpha,\beta,\gamma}\frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)}\ \frac{t^{\alpha+\frac{1}{2}\nu-\epsilon}}{\Gamma(\alpha+\nu-\epsilon+1)}$$

$$\times{}_3F_3\begin{bmatrix}\alpha,\ \alpha-\delta+1,\ \alpha-\lambda+1;\ -t\\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha+\nu-\epsilon+1\end{bmatrix}dt$$

$$\doteqdot\frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\sum_{\alpha,\beta,\gamma}\frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)}p^{\alpha-\epsilon+1}\cdot{}_3F_2\begin{bmatrix}\alpha,\ \alpha-\delta+1,\ \alpha-\lambda+1\\ \alpha-\beta+1,\ \alpha-\gamma+1\end{bmatrix};\ -p\Bigg]$$

$$=\frac{1}{p^{\epsilon-1}}\cdot{}_3F_2[\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda;\ -1/p]\doteqdot\frac{x^{\epsilon-1}}{\Gamma(\epsilon)}\cdot{}_3F_3[\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \epsilon;\ -x]. \quad (9)$$

Hence

$$\frac{\Gamma(\delta)\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_0^\infty x^{\frac12\nu}J_\nu[2(xt)^{\frac12}]\sum_{\alpha,\beta,\gamma}\frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\alpha)}{\Gamma(\delta-\alpha)\Gamma(\lambda-\alpha)}\,\frac{t^{\alpha+\frac12\nu-\epsilon}}{\Gamma(\alpha+\nu-\epsilon+1)}$$

$$\times\,{}_3F_3[\alpha,\ \alpha-\delta+1,\ \alpha-\lambda+1\,;\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha+\nu-\epsilon+1\,;\ -t]\,dt$$

$$=\frac{x^{\epsilon-1}}{\Gamma(\epsilon)}\,{}_3F_3[\alpha,\ \beta,\ \gamma\,;\ \delta,\ \lambda,\ \epsilon\,;\ -x]. \qquad (9')$$

Writing $\frac12 x^2$ for x and $\frac12 t^2$ for t and simplifying a bit, we get the result

$$\int_0^\infty J_\nu(xt)(xt)^{\frac12}\psi(t)\,dt=\phi(x)$$

5. Let us put $\beta=\frac12(\alpha+m+1)$, $\gamma=\frac12(\alpha+n+1)$, $\delta=\frac12(\alpha-m+1)$, $\lambda=\frac12(\alpha-n+1)$ and $\epsilon=\frac12(\alpha+\nu+1)$; where m and n are positive integers. Then

$$\int_0^\infty J_\nu(xt)(xt)^{\frac12}t^{\alpha-\frac12}\,{}_3F_3\left[\begin{matrix}\alpha,\ \frac12(\alpha+m+1),\ \frac12(\alpha+n+1)\,;\\ \frac12(\alpha-m+1),\ \frac12(\alpha-n+1),\ \frac12(\alpha+\nu+1)\end{matrix}\ -\tfrac12 t^2\right]dt$$

$$=(-1)^{m+n}x^{\alpha-\frac12}\,{}_3F_3\left[\begin{matrix}\alpha,\ \frac12(\alpha+m+1),\ \frac12(\alpha+n+1)\,;\\ \frac12(\alpha-m+1),\ \frac12(\alpha-n+1),\ \frac12(\alpha+\nu+1)\end{matrix}\ -\tfrac12 x^2\right] \qquad (10)$$

which shows that

$$x^{\alpha-\frac12}\,{}_3F_3\left[\begin{matrix}\alpha,\ \frac12(\alpha+m+1),\ \frac12(\alpha+n+1)\,;\ -\frac12 x^2\\ \frac12(\alpha-m+1),\ \frac12(\alpha-n+1),\ \frac12(\alpha+\nu+1)\end{matrix}\right]$$

is self-reciprocal or skew self-reciprocal in the Hankel Transform of order ν, provided $\mathbf{R}(\alpha)>-\frac12$, and $\mathbf{R}(\alpha+\nu)>-1$. A further generalisation of this result is under investigation.*

In conclusion I wish to express my indebtedness to Dr S. C. Mitra for his help and guidance in the preparation of the paper.

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* Art. 4 is not necessary to prove in view of Hankel's Theorem. The generalised result is coming out in the *Jour. Indian Math. Soc.*

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RANDOM DISTANCES WITHIN A RECTANGLE AND BETWEEN TWO RECTANGLES

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Introduction

The distributions of random distances within a figure, and between two figures in a plane, have important applications in statistics, specially in problems of *topographic variation* (*Vide*: B. Matérn, 1947, p. 123; B. Ghosh, 1949, p 20; M. N. Ghosh, 1949, p. 85; Garwood, 1947; Armitage, 1949). Such distributions have been worked out within a rectangle*, and between two rectangles with similar orientations, and the results for some particular cases have been briefly recorded in earlier notes (B. Ghosh, 1943*a*, 1943*b*) As some authors have required the results for other cases not covered by the earlier notes, the general method of evaluating such distributions will be described here, following which any required particular case can be tackled.

A rigorous statement of the problems may be given first. Let us consider a rectangle (which is usually a *sample-unit* in *area sampling*) with sides equal to a and b along x and y axes ($x = 0$ to a; $y = 0$ to b). The rectangle, as part of the statistical field, is composed of "points", each "point" being the centre of a very small square called the *basic cell* (*vide* B. Ghosh, 1949, pp. 13 14); all the square cells are of the same size and have their sides parallel to x and y axes. A cell being extremely small compared with the rectangle, for all practical purposes the co-ordinates, x and y, of the "points" may be regarded as varying continuously with their joint *probability density function* (p. d. f.) given by $f(x, y) = 1/(ab)$ for $x = 0$ to a, and $y = 0$ to b. Two "points" P_1 and P_2 are located randomly and independently (in the stochastic sense) within the rectangle, with their co-ordinates denoted by (x_1, y_1) and (x_2, y_2) respectively. Consider the interval-vector, I, connecting the two points, P_1 and P_2, with its components along x and y given by (x_2-x_1) and (y_2-y_1). It is required to find out the p. d. f., $f(R)$, of R, the length of I, defined by $R = +\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2\}}$. This problem may be called briefly the "one rectangle" problem.

Next, we consider two rectangles with similar orientations defined by say, $x = 0$ to a and $y = 0$ to b for the first rectangle, and $x = (a+e)$ to $(a+c+e)$ and $y = (b+f)$ to $(b+d+f)$ for the second. Two points P_1 and P_2 are located randomly in the two rectangles, P_1 in the first rectangle and P_2 in the second. The p. d. f. of (x_1, y_1), the

* Some particular results covered by (Ghosh 1943a) have been given later by other authors; see also Santalo (1947)

co-ordinates of P_1, is given by $f(x_1, y_1) = 1/(ab)$, for $x_1 = 0$ to a and $y_1 = 0$ to b; similarly for P_2 we have $f(x_2, y_2) = 1/(cd)$, for $x_2 = (a+e)$ to $(a+c+e)$ and $y_2 = (b+f)$ to $(b+d+f)$. In this problem also we require $f(R)$, R being the length of the interval I connecting P_1 and P_2. This may be called the "two rectangles" problem.

Problem of One Rectangle

Let us define $X = |x_1 - x_2|$ and $Y = |y_1 - y_2|$, so that $R = +\sqrt{(X^2+Y^2)}$. Since P_1 and P_2 are independent, and for any point, x, y are independent, obviously X, Y also will be independently distributed. Theorem 1 regarding the p. d. f. of X is stated below, the proof of which, being quite simple, is omitted here.

Theorem 1. If x_1 and x_2 are two stochastically independent and random values from the same distribution whose p. d. f. is given by $f(x) = 1/a$ for $x = 0$ to a, then the p. d. f. of $X = |x_1 - x_2|$ is given by $f(X) = 2(a-X)/a^2$, for $X = 0$ to a.

Here X can also be regarded as the distance between two random points in a straight line of length a. Similarly we get the p. d. f. of Y. Since X, Y are independent their joint p. d. f. is given by

$$f(X, Y) = \frac{4(a-X)(b-Y)}{a^2b^2}, \text{ for } X = 0 \text{ to } a \text{ and } Y = 0 \text{ to } b. \qquad (1)$$

We now transform the variables (X, Y) to (R, θ), where $\tan\theta = Y/X$, (R already defined). Within the effective ranges $X = 0$ to a and $Y = 0$ to b, $f(X, Y)dXdY$ is therefore transformed to

$$f(R, \theta)dRd\theta = \frac{4R}{a^2b^2}(a - R\cos\theta)(b - R\sin\theta)dRd\theta. \qquad (2)$$

Integrating (2) over θ, between the appropriate limits, θ_1 (lower) and θ_2 (upper), we get the p. d. f. of R, $f(R) = (4R/a^2b^2)\phi(R)$, where

$$\phi(R) = ab(\theta_2 - \theta_1) + aR(\cos\theta_2 - \cos\theta_1) - bR(\sin\theta_2 - \sin\theta_1) - \frac{R^2}{4}(\cos 2\theta_2 - \cos 2\theta_1). \qquad (3)$$

Care is necessary in finding the values θ_1 and θ_2. Consider the "effective rectangle," $X = 0$ to a, $Y = 0$ to b, in the (X, Y)-plane. (In this case we can assume $a \geq b$, without any loss of generality, since it is merely a matter of choosing the x and y axes suitably). From inspection it will be seen that the "effective rectangle" can be divided in terms of R into three convenient ranges, (i) $R = 0$ to b, (ii) $R = b$ to a, and (iii) $R = a$ to $\sqrt{(a^2+b^2)}$. Further, in range (i) $\theta_1 = 0$, $\theta_2 = \frac{1}{2}\pi$; in range (ii) $\theta_1 = 0$, $\theta_2 = \sin^{-1}(b/R)$; and in range (iii) $\theta_1 = \cos^{-1}(a/R)$, $\theta_2 = \sin^{-1}(b/R)$. Putting these values in equation (3) and simplifying, we get the following theorem.

Theorem 2. The p. d. f. of R, the distance between two independent random points in a rectangle with sides a and b, $(a \geq b)$, is given by $f(R) = (4R/a^2b^2)\phi(R)$, where

$$\phi(R) = \tfrac{1}{2}\pi ab - aR - bR + \tfrac{1}{2}R^2, \text{ for } R = 0 \text{ to } b;$$

$$\phi(R) = ab\sin^{-1}(b/R) + a\sqrt{(R^2-b^2)} - aR - \tfrac{1}{2}b^2, \text{ for } R = b \text{ to } a;$$

and $\phi(R) = ab\{\sin^{-1}(b/R) - \cos^{-1}(a/R)\} + a\sqrt{(R^2-b^2)}$

$$+b\sqrt{(R^2-a^2)} - \tfrac{1}{2}(R^2+a^2+b^2), \text{ for } R = a \text{ to } \sqrt{(a^2+b^2)}.$$

The first four moments of this distribution are given in the appendix (eqn. 15). At any transitional value of R, $f(R)$ and df/dR have the same values in both the adjacent ranges, while the values of d^2f/dR^2 are not so. When $a = b$ (square), the second range in Theorem 2, $R = b$ to a, is non-existent.

Problem of Two Rectangles

Here also we define $X = |x_1 - x_2|$, $Y = |y_1 - y_2|$. The joint p. d. f. of x_1 and x_2 is given by $f(x_1, x_2) = 1/(ac)$, for $x_1 = 0$ to a and $x_2 = (a+e)$ to $(a+c+e)$. Transforming (x_1, x_2) to (u, v), with $u = (x_2 - x_1)$, $v = x_1$, we have $f(u, v) = 1/(ac)$ in the effective ranges for x_1 and x_2. Integrating $f(u, v)$ over v, between the appropriate limits, v_1 (lower) and v_2 (upper), we get,

$$f(u) = \frac{v_2 - v_1}{ac}, \text{ in the effective ranges of } x_1,\ x_2. \tag{4}$$

From inspection of the "effective rectangle," $x_1 = 0$ to a, $x_2 = (a+e)$ to $(a+c+e)$, in the (x_1, x_2)-plane we have three convenient ranges for u, (i) $u = e$ to $(e+g)$, (ii) $u = (e+g)$ to $(e+h)$, and (iii) $u = (e+h)$ to $(e+g+h)$, where g stands for the smaller of the two values, a and c, and h for the greater. (If $a = c$, then $g = h = a$). If $a \geq c$, it can be shown from inspection of (x_1, x_2)-diagram that in range (i) $v_1 = (e+h-u)$, $v_2 = h$, so that $(v_2 - v_1) = (u-e)$; in range (ii) $v_1 = (e+h-u)$, $v_2 = (e+g+h-u)$, and $(v_2 - v_1) = g$; and in range (iii) $v_1 = 0$, $v_2 = (e+g+h-u)$, and $(v_2 - v_1) = (e+g+h-u)$. If alternatively $a < c$, the values of $(v_2 - v_1)$ in the three ranges will remain unaltered in terms of g and h. Since $X = u$, we have the following result from eqn. (4):—

Theorem 3. If x_1, x_2 are independently distributed with p. d. f.'s, $f(x_1) = 1/a$, for $x_1 = 0$ to a, and $f(x_2) = 1/c$, for $x_2 = (a+e)$ to $(a+c+e)$, the p. d. f. of $X = |x_1 - x_2|$ is given by $f(X) = \phi(X)/(ac)$, where $\phi(X) = (X-e)$ for $X = e$ to $(e+g)$; $\phi(X) = g$, for $X = (e+g)$ to $(e+h)$; and $\phi(X) = (g+h+e-X)$, for $X = (e+h)$ to $(e+g+h)$. (The symbols, g, h, have been explained before).

Here X may also be regarded as the distance between two random points P_1, P_2, selected respectively from two straight lines of lengths a and c, one line lying on the extension of the other, with their nearest points separated by a distance, e. Or, the two straight lines of lengths a and c, parallel to the x-axis, may have their nearest points separated by a distance e along x, and f along y; here X will represent the absolute value of the x-component of I, the interval connecting P_1 and P_2, while the y-component of I will be a constant equal to f.

Corollary 1 to theorem 3. If $a = c$, $f(X) = \phi(X)/a^2$, where $\phi(X) = (X-e)$, for $X = e$ to $(a+e)$; and $\phi(X) = (2a+e-X)$, for $X = (a+e)$ to $(2a+e)$.

In case the effective ranges of x_1 and x_2 are overlapping, we have to formally take e as negative. Further X will not always be $(+u)$, but sometimes $(-u)$ as well, and so

some care has to be taken in changing over from u to X. A case of complete overlapping with $a = c$, and $e = -a$ (formally), is of some practical interest, and can be derived from corollary 1.

Corollary 2 to theorem 3. In case of "complete overlapping" ($a = c$, $e = -a$), $f(X) = 2(a-X)/a^2$, for $X = 0$ to a. (This result may be compared with theorem 1).

Returning to the general problem of "two rectangles," $f(Y)$ can be derived from theorem 3, by replacing a, c, e, g, h with b, d, f, p, q respectively, (p, q) being defined with respect to (b, d) in the same manner as (g, h) with respect to (a, c). Further $f(X, Y) = f(X)f(Y)$. Defining R, θ as in the problem of "one rectangle," we transform $f(X, Y)dXdY$ to $f(R, \theta)dRd\theta$, and then integrate out θ between appropriate limits θ_1 and θ_2, to get $f(R)$. The problem here is, however, more complicated than the "one rectangle" problem. The "effective rectangle" $X = e$ to $(e+g+h)$, $Y = f$ to $(f+p+q)$ in the (X, Y)-diagram is divided into 9 "compartments" formed by the combinations of three effective ranges of X and three ranges of Y, whereas in the "one-rectangle problem" there is only one such "compartment". Here as we change θ, for a given value of R, we may have to pass through several compartments, and the function $f(R, \theta)$ and the lower and upper limits of θ will be different in different compartments. These points will be clear from the study of the (X, Y)-diagram. In the most general case, there will be 16 transitional points in the "effective rectangle" given by the combinations of $X = e$, $(e+g)$, $(e+h)$, $(e+g+h)$ and $Y = f$, $(f+p)$, $(f+q)$, $(f+p+q)$. By considering the *iso-R* lines in the (X, Y)-diagram it will be seen that there will be 16 transitional values of R, passing through the 16 transitional points, and so there will be 15 effective ranges of R instead of only 3 such ranges of the "one-rectangle" problem. Though any particular case with known values of a, b, c, d, e, f can be always solved, it is no use attempting a general solution, which will be too much involved. Some special cases are discussed below to illustrate the method.

Equal Squares with Common Diagonal Line

Here $a = b = c = d$, $e = f$. The joint p. d. f. $f(X, Y) = f(X)f(Y)$ can be written down from corollary 1 to theorem 3. The "effective rectangle" in the (X, Y)-diagram is divided into four compartments: the first $C(I)$ is defined as $X = e$ to $(e+a)$, $Y = e$ to $(e+a)$, $C(II)$ is $X = (e+a)$ to $(e+2a)$, $Y = e$ to $(e+a)$; $C(III)$ is $X = e$ to $(e+a)$, $Y = (e+a)$ to $(e+2a)$; and $C(IV)$ is $X = (e+a)$ to $(e+2a)$, $Y = (e+a)$ to $(e+2a)$. Though there are 9 transitional points, $X = e$, $(e+a)$, $(e+2a)$ with $Y = e$, $(e+a)$, $(e+2a)$, because of symmetry in X and Y, we have only 6 transitional values of R, given by $R_1 = \sqrt{2}e$, $R_2 = \sqrt{\{e^2+(e+a)^2\}}$, $R_3 = \sqrt{2}(e+a)$, $R_4 = \sqrt{\{e^2+(e+2a)^2\}}$, $R_5 = \sqrt{\{(e+a)^2 + (e+2a)^2\}}$, and $R_6 = \sqrt{2}(e+2a)$. Thus there are only 5 effective ranges of R, (R_1 to R_2, R_2 to R_3, . . ., R_5 to R_6) in this case.

First Range ($R = R_1$ to R_2). In this range the point in the (X, Y)-diagram is solely confined to $C(I)$, in which $f(X, Y) = (X-e)(Y-e)/a^4$, and so $f(R, \theta) = R(R\cos\theta - e)(R\sin\theta - e)/a^4$. The limits of θ are given by $\theta_1 = \sin^{-1}(e/R)$ and $\theta_2 = \cos^{-1}(e/R)$,

as will be clear from inspection of the (X, Y)-diagram. Integrating $f(R, \theta)$ over θ, from θ_1 to θ_2, and simplifying, we have

$$f(R) = \frac{R}{a^4}\left\{e^2\left(\cos^{-1}\frac{e}{R} - \sin^{-1}\frac{e}{R}\right) - 2e\sqrt{(R^2-e^2)} + \tfrac{1}{2}R^2 + e^2\right\}. \tag{5}$$

Second Range ($R = R_2$ *to* R_3). This range represents the strip of the "effective rectangle" in the (X, Y)-diagram between the two circular lines $R = R_2$, $R = R_3$. It will be seen that for a given R as one increases θ, one has to pass through $C(II)$, $C(I)$ and $C(III)$ successively.

In $C(II)$, we have $f(R, \theta) = R(2a+e-R\cos\theta)(R\sin\theta - e)/a^4$; further the limits of θ are $\theta_1 = \sin^{-1}(e/R)$, $\theta_2 = \cos^{-1}(e+a)/R$. Integrating $f(R, \theta)$ over θ, from θ_1 to θ_2, we have,

$$R[-\tfrac{3}{2}a^2 - e^2 - 2ae + e\sqrt{\{R^2-(a+e)^2\}} + (2a+e)\sqrt{(R^2-e^2)}$$
$$- e(2a+e)\{\cos^{-1}(a+e)/R - \sin^{-1}(e/R)\}]/a^4 = \alpha, \text{ say.} \tag{6}$$

Next, in $C(I)$, $f(R, \theta) = R(R\cos\theta - e)(R\sin\theta - e)/a^4$, and the limits $\theta_1 = \cos^{-1}(a+e)/R$, $\theta_2 = \sin^{-1}(a+e)/R$. (These limits θ_1, θ_2 are different from the limits for the same compartment, $C(I)$, in the first range, R_1 to R_2). Integrating $f(R, \theta)$ over θ, between θ_1 and θ_2, we have,

$$R[-\tfrac{1}{2}R^2 + a^2 - e^2 + 2e\sqrt{\{R^2-(a+e)^2\}}$$
$$+ e^2\left\{\sin^{-1}\left(\frac{a+e}{R}\right) - \cos^{-1}\left(\frac{a+e}{R}\right)\right\}\Big]\Big/ a^4 = \beta, \text{ say.} \tag{7}$$

Finally, in $C(III)$, integrating $f(R, \theta) = R(R\cos\theta - e)(2a+e-R\sin\theta)/a^4$ over θ between $\theta_1 = \sin^{-1}(a+e)/R$ and $\theta_2 = \cos^{-1}(e/R)$, we have,

$$R\left[-e^2 - \tfrac{3}{2}a^2 - 2ae - \frac{R^2}{2} + (2a+e)\sqrt{(R^2-e^2)} + e\sqrt{\{R^2-(a+e)^2\}}\right.$$
$$\left.- e(2a+e)\left\{\cos^{-1}\left(\frac{e}{R}\right) - \sin^{-1}\left(\frac{a+e}{R}\right)\right\}\right]\Big/ a^4 = \gamma, \text{ say.} \tag{8}$$

Now adding together the contributions from $C(II)$, $C(I)$ and $C(III)$ represented by α, β and γ (eqns. 6, 7, 8), we have, in the range $R = R_2$ to R_3,

$$f(R) = R\left[2e(a+e)\left\{\sin^{-1}\left(\frac{a+e}{R}\right) - \cos^{-1}\left(\frac{a+e}{R}\right)\right\}\right.$$
$$+ e(2a+e)\left\{\sin^{-1}\left(\frac{e}{R}\right) - \cos^{-1}\left(\frac{e}{R}\right)\right\} + 4e\sqrt{\{R^2-(a+e)^2\}}$$
$$\left.+ (4a+2e)\sqrt{(R^2-e^2)} - \tfrac{3}{2}R^2 - 2a^2 - 4ae - 3e^2\right]\Big/ a^4. \tag{9}$$

Proceeding in this manner we can work out the expressions for $f(R)$ in the other three ranges of R also. Now we shall state without proof the results for $e = 0$, as that case is of some practical importance.

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Equal Squares with Corner-point Contact

Here $a = b = c = d$, $e = f = 0$. The p. d. f. $f(R)$ is given in the following five ranges, as $f(R) = R\phi(R)/a^4$, where

$$\phi(R) = \tfrac{1}{2}R^2, \text{ for } R = 0 \text{ to } a;$$

$$\phi(R) = -\tfrac{3}{2}R^2 + 4aR - 2a^2, \text{ for } R = a \text{ to } \sqrt{2}a;$$

$$\phi(R) = 4a^2\{\cos^{-1}(a/R) - \sin^{-1}(a/R)\} - 8a\sqrt{(R^2-a^2)} + \tfrac{1}{2}R^2 + 4aR + 2a^2, \text{ for } R = \sqrt{2}a \text{ to } 2a;$$

$$\phi(R) = 4a^2\{\cos^{-1}(a/R) - \sin^{-1}(a/R)\} - 8a\sqrt{(R^2-a^2)} + \tfrac{3}{2}R^2 + 6a^2, \text{ for } R = 2a \text{ to } \sqrt{5}a;$$

and
$$\phi(R) = 4a^2\{\sin^{-1}(2a/R) - \cos^{-1}(2a/R)\} + 4a\sqrt{(R^2-4a^2)} - \tfrac{1}{2}R^2 - 4a^2, \text{ for } R = \sqrt{5}a \text{ to } 2\sqrt{2}a. \quad (10)$$

It can be shown that at any transitional value of R, $f(R)$ has got the same values in both the adjacent ranges. The mean value of R comes out as $1.473a$ approximately (*vide* appendix).

Equal Adjacent Squares

This case is also of practical importance, and will be briefly discussed here, without details of proof. Suppose the squares are adjacent in the x-direction, so that we may put $a = b = c = d$, $e = 0, f = -a$. The p. d. f. of X, $f(X)$ is given by corollary 1 to theorem 3, and $f(Y)$ by corollary 2. In the (X, Y)-diagram we have only two compartments $C(I)$ and $C(II)$, $C(I)$ being given by $X = 0$ to a, $Y = 0$ to a, and $C(II)$ by $X = a$ to $2a$, $Y = 0$ to a; there are six transitional points $X = 0, a, 2a$ combined with $Y = 0, a$, and only five transitional values of $R = 0, a, \sqrt{2}a, 2a, \sqrt{5}a$. The p. d. f. of R, is given in four ranges, as $f(R) = 2R\phi(R)/a^4$, where

$$\phi(R) = aR - \tfrac{1}{2}R^2, \text{ for } R = 0 \text{ to } a;$$

$$\phi(R) = 2a^2\cos^{-1}(a/R) - 2a\sqrt{(R^2-a^2)} - 2aR + R^2 + \tfrac{3}{2}a^2, \text{ for } R = a \text{ to } \sqrt{2}a;$$

$$\phi(R) = 2a^2\sin^{-1}(a/R) + 2a\sqrt{(R^2-a^2)} - 2aR - \tfrac{1}{2}a^2, \text{ for } R = \sqrt{2}a \text{ to } 2a;$$

and
$$\phi(R) = 2a^2\{\sin^{-1}(a/R) - \cos^{-1}(2a/R)\} + 2a\sqrt{(R^2-a^2)} + a\sqrt{(R^2-4a^2)} - \tfrac{1}{2}R^2 - \tfrac{5}{2}a^2, \text{ for } R = 2a \text{ to } \sqrt{5}a. \quad (11)$$

For any transitional value of R, $f(R)$ has got the same value in both the adjacent ranges. The mean R comes out as $1.088a$ approximately (*vide* appendix).

Indirect Methods

A method has been developed for evaluating $f(R)$ indirectly for some new cases, with the help of already known expressions of $f(R)$ for some other cases; a simple illustration of this method is given below. Consider a rectangle with adjacent sides

equal to a and $2a$, in which two independent random points P_1 and P_2 are located, and let the p. d. f., $f(R)$, be denoted by f_1. Now considering the rectangle as made up of two equal adjacent squares of side a, it will be seen that the probability of P_1 and P_2 belonging to the same square is equal to the probability of P_1 and P_2 belonging to different squares, each probability being $\frac{1}{2}$. So, if f_2 denotes the p. d. f., $f(R)$, within a square of side a, and f_3 denotes the p. d. f., $f(R)$, between two equal adjacent squares of side a, it will be clear with a little thought that f_1 will be equal to $\frac{1}{2}(f_2+f_3)$. So of these three functions, f_1, f_2, f_3, if any two are already known, the third can be easily found out in this indirect manner. For this particular example, of course, we can easily find out f_1 and f_2 from theorem 2, and f_3 is given by eqn. (11), and so we can verify the relation $f_1 = \frac{1}{2}(f_2+f_3)$. Denoting the mean R for the three distributions f_1, f_2, f_3 by M_1, M_2, M_3, we also have $M_1 = \frac{1}{2}(M_2+M_3)$, which relation can also be easily verified, as from the appendix we find the approximate values of M_1, M_2, M_3 as $0.804a$, $0.521a$ and $1.088a$.

For plane figures of other shapes (non-rectangular) it will not usually be possible to derive the expressions for $f(R)$ theoretically. But, if necessary, approximate nature of the distribution $f(R)$ can be ascertained by empirical methods, *e.g.* experimental sampling.

Appendix

The k-th moment about the origin, α_k, is the integral of $\{R^k f(R)\}$ over the whole range of R. For evaluating these moments for both the problems of "one rectangle" and "two rectangles" the following integrals will be required:

$$\int R^n \sin^{-1}\left(\frac{m}{R}\right) dR,\ \int R^n \cos^{-1}\left(\frac{m}{R}\right) dR, \text{ and } \int R^n \sqrt{(R^2-m^2)}dR,$$

with positive integral values of n. If we require α_k for $k = 0, 1, 2, 3, 4$ only, the necessary values of n are 1, 2, 3, 4, 5. By integration by parts, we have

$$\int \sin^{-1}\left(\frac{m}{R}\right) R^n dR = \frac{1}{n+1} R^{n+1} \sin^{-1}\left(\frac{m}{R}\right) + \frac{m}{n+1}\int \frac{R^n dR}{\sqrt{(R^2-m^2)}},$$

$$\int \cos^{-1}\left(\frac{m}{R}\right) R^n dR = \frac{1}{n+1} R^{n+1} \cos^{-1}\left(\frac{m}{R}\right) - \frac{m}{n+1}\int \frac{R^n dR}{\sqrt{(R^2-m^2)}}. \qquad (12)$$

So ultimately we require integrals of the form

$$I_n = \int \frac{R^n dR}{\sqrt{(R^2-m^2)}}, \quad \text{and} \quad J_n = \int R^n \sqrt{(R^2-m^2)}dR,$$

for $n = 1, 2, 3, 4, 5$.

By successive integration by parts, and putting $P = \sqrt{(R^2-m^2)}$, we have, for odd values of n,

$$I_1 = P;\ I_3 = \tfrac{1}{3}P^3 + m^2P;\ I_5 = \tfrac{1}{5}P^5 + \tfrac{2}{3}m^2P^3 + m^4P;$$

$$J_1 = \tfrac{1}{3}P^3;\ J_3 = \tfrac{1}{5}P^5 + \tfrac{1}{3}m^2P^3;\ J_5 = \tfrac{1}{7}P^7 + \tfrac{2}{5}m^2P^5 + \tfrac{1}{3}m^4P^3. \qquad (13)$$

We further put $Q = \cosh^{-1}(R/m)$; of the two roots of $\cosh^{-1}(R/m)$, the principal value, $[\log_e\{R+\sqrt{(R^2-m^2)}\}-\log_e m]$, is to be taken. Now by successive integration by parts, we have, for even values of n,

$$I_2 = \tfrac{1}{2}RP+\tfrac{1}{2}m^2Q\,; \quad I_4 = \tfrac{1}{8}RP(2R^2+3m^2)+\tfrac{3}{8}m^4Q\,;$$

$$J_2 = \tfrac{1}{8}RP(2R^2-m^2)-\tfrac{1}{8}m^4Q\,; \quad J_4 = \tfrac{1}{48}RP(8R^4-2m^2R^2-3m^4)-\tfrac{1}{16}m^6Q. \tag{14}$$

Using these relations (12), (13), (14), we can work out the values of α_k for different cases.

"One-Rectangle" Case. For $f(R)$ given in theorem 2, we have, putting $M = \sqrt{(a^2+b^2)}$, for even values of k, $\alpha_0 = 1$ (as it should be),

$$\alpha_2 = \tfrac{1}{6}M^2\,; \quad \alpha_4 = \tfrac{1}{15}a^4+\tfrac{1}{18}a^2b^2+\tfrac{1}{15}b^4\,;$$

and for odd values of k,

$$\alpha_1 = \frac{1}{6}\left\{\frac{b^2}{a}\cosh^{-1}\left(\frac{M}{b}\right)+\frac{a^2}{b}\cosh^{-1}\left(\frac{M}{a}\right)\right\}+\frac{1}{15}\left(\frac{a^3}{b^2}+\frac{b^3}{a^2}\right)-\frac{1}{15}M\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}-3\right);$$

$$\alpha_3 = \frac{1}{20}\left\{\frac{a^4}{b}\cosh^{-1}\left(\frac{M}{a}\right)+\frac{b^4}{a}\cosh^{-1}\left(\frac{M}{b}\right)\right\}+\frac{2}{105}\left(\frac{a^5}{b^2}+\frac{b^5}{a^2}\right)$$

$$-M\left\{\frac{2}{105}\left(\frac{a^4}{b^2}+\frac{b^4}{a^2}\right)-\frac{5}{84}M^2\right\}. \tag{15}$$

With known values of a and b, the values of mean, variance, skewness (γ_1) and kurtosis (γ_2) can be worked out from eqns (15). For example, the approximate values of α_1 (mean) for $a = b$ and $a = 2b$ are given by 0 521a and 0.402a respectively. It may be noted here that some of the numerical values for these measures have been wrongly printed in (B. Ghosh 1943a).

"Two Rectangles" Case. For the special cases of "equal squares with corner-point contact" (eqn. 10), and "equal adjacent squares" (eqn. 11), it has been verified that $\alpha_0 = 1$; the approximate values of the mean distance (α_1) are 1.473a and 1.088a respectively in the two cases.

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ON THE MAXIMUM MODULUS OF AN INTEGRAL FUNCTION

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1. Introduction. Let $f(z) = \sum_{0}^{\infty} a_n z^n$ be an Integral function of order ϱ. Let $m(r)$ denote the maximum term, of the series for $|z| = r$ and $N(r)$ the rank of this term, then Dr. S. M. Shah [(1950), p. 21 ; (1950) p. 112-113] has proved the following results :

(i) If $f(z)$ be a function of finite order, then

$$\overline{\lim_{r\to\infty}} \left\{\frac{\log M(r)}{N(r) \log r}\right\} \leq 1 \tag{1.1}$$

the equality sign holding if

$$\lim_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = 1$$

(ii) If

$$\log\log m(r) = \{1 + o(1)\} \log\log r$$

for a sequence of values of r tending to infinity, then

$$\overline{\lim_{r\to\infty}} \frac{\log m(r)}{N(r) \log r} = 1 \tag{1.2}$$

(iii) If $f(z)$ is of any order,

$$\overline{\lim_{r\to\infty}} \frac{\log m(r)}{N(r) \log r} \leq 1. \tag{1.3}$$

The object of this note is to give a simple proof of (1.1) for Integral functions of finite order. Two other similar results involving $A(r)$ and $M^{(s)}(r)$, where $A(r)$ and $M^{(s)}(r)$ are the maximum real part and maximum modulus of $f(z)$ and $f^{(s)}(z)$ for $|z| = r$ respectively, have been given in Theorems II and III.

2. Theorem I. *If $f(z) = \sum_{0}^{\infty} a_n z^n$ is an Integral function of order $\varrho(0 \leq \varrho < \infty)$, then*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{N(r) \log r} = \overline{\lim_{r\to\infty}} \frac{\log m(r)}{N(r) \log r} \leq 1, \tag{2.1}$$

where $M(r) = \max_{|z|=r} |f(z)|$, and $m(r)$ and $N(r)$ are the maximum term and the rank of the maximum term respectively for $|z| = r$.

Proof: We have (Valiron, 1923 p. 34) for an Integral function, $f(z)$, of order $\varrho(0 \leq \varrho < \infty)$,

$$m(r) < M(r) < m(r) r^{\rho+\epsilon}. \tag{2.2}$$

On taking logarithm and dividing by $N(r) \log r$, which is positive and non-decreasing, and taking the limit for r tending to infinity, we get

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{N(r)\log r} = \overline{\lim_{r\to\infty}} \frac{\log m(r)}{N(r)\log r} \tag{2.3}$$

Now, consider the result (Valiron, p. 31).

$$\log m(r) = \log m(r_0) + \int_{r_0}^{r} \frac{N(x)}{x}\,dx \qquad \text{for } 0 < r_0 < r. \tag{2.4}$$

Since $N(r)$ is a non-deceasing positive function of r, therefore

$$\log m(r) < \log m(r_0) + N(r)\log r/r_0,$$

whence

$$\log m(r) \leq \left[\frac{\log r - \log r_0}{\log r} + \theta\,\frac{\log m(r_0)}{N(r)\,\log r}\right] N(r)\log r, \quad 0 < \theta < 1$$

and on taking limit, we see

$$\lim_{r\to\infty} \frac{\log m(r)}{N(r)\log r} \leq 1. \tag{2.5}$$

Hence, the result follows from (2 3) and (2 5)

3. Theorem II. *If* $f(z) = \sum_0^\infty a_n z^n$ *is an Integral function of order* $\varrho(0 \leq \varrho < \infty)$, *then*

$$\overline{\lim_{r\to\infty}} \frac{\log A(r)}{N(r)\log r} \leq 1, \tag{3.1}$$

where $A(r)$ *is the maximum real part of* $f(z)$ *for* $|z| = r$.

The proof of this theorem easily follows from Theorem I and the result of Valiron (p. 34)

$$A(r) < M(r) < A(r) r^{\rho+\epsilon}$$

4 Theorem III. *If* $f(z) = \sum_0^\infty a_n z^n$ *is an Integral function of order* $\varrho(0 \leq \varrho < \frac{1}{2})$ *then*

$$\overline{\lim_{r\to\infty}} \frac{\log M^{(s)}(r)}{N(r)\log r} \leq \overline{\lim_{r\to\infty}} \frac{\log M(r)}{N(r)\log r} \leq 1 \tag{4.1}$$

$|f^{(s)}(z)|$ *where* $M^{(s)}(r) = \max_{|z|=r} |f^{(s)}(z)|$, $f^{(s)}(z)$ *is the s-th derivative of* $f(z)$.

The proof of this theorem easily follows from Theorem II (Bose 1946, p. 79) and Theorem I of this paper.

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SOME PROBLEMS OF ELASTIC PLATES CONTAINING CIRCULAR HOLES—I

By

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1. *Introduction*

Owing to great practical importance in Engineering application, the problems of circular holes in a stressed plate have been discussed by a large number of authors. A general solution in bipolar coordinates of the stress and strain for a plate having two circular boundaries has been given by Jeffery (1921) who has also given the solution for a semi-infinite plate, with an unstressed circular hole under tension parallel to its straight edge. The solutions for an infinite plate containing two equal unstressed circular holes were obtained (1) by Ghosh (1939) when the plate is subjected to (*i*) a uniform tension in the direction of the line of centres and (*ii*) a uniform shear in the plane of the plate and (2) by Weinel (1937) when the plate is subjected to a uniform tension perpendicular to the line of centres. Recently Ling (1948*a*, 1948*b*) has discussed the cases of an infinite plate containing two equal circular holes under the action of (*a*) an all-round tension T, (*b*) a longitudinal tension T, and (*c*) a transverse tension T, when the circular holes are tangential to each other as also when the holes overlap. In the present paper solutions in bipolar coordinates, are given for a plate of infinite size with two unstressed equal circular holes subjected to the action of (*I*) a couple of moment M and (*II*) a centre of pressure midway between them.

2. *Mathematical treatment*

Let us take the curvilinear coordinates defined by

$$\alpha + i\beta = \log \frac{x + i(y+a)}{x + i(y-a)}, \tag{1}$$

where x, y are cartesian coordinates and a is a positive length.

Solving for x and y we have

$$x = \frac{a \sin\beta}{\cosh\alpha - \cos\beta}, \quad y = \frac{a \sinh\alpha}{\cosh\alpha - \cos\beta}, \tag{2}$$

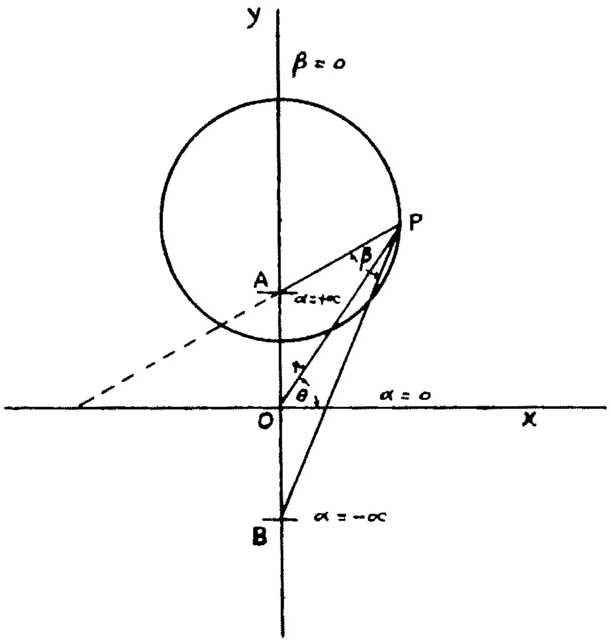
so that from the relation

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2$$

we get,

$$h = (\cosh\alpha - \cos\beta)/a. \tag{3}$$

The curves $\alpha =$ constant are a set of co-axial circles having the two poles $A(0, a)$ and $B(0, -a)$ for limiting points. The circles corresponding to positive values of α lie



above the axis of x, while those corresponding to negative values lie below. The axis of x is given by $\alpha = 0$. The radius of a circle α, is $a\operatorname{cosech}\alpha$, while the coordinates of its centre are $(0, a\coth\alpha)$.

The curves $\beta =$ constant are a system of circles passing through the limiting points A, B and intersecting the other set of circles orthogonally. β is positive on the right-hand side of the y-axis and negative on the left-hand side. While on the y-axis $\beta = 0$, except on the segment AB where $\beta = \pm\pi$.

At infinity $\alpha = 0$, $\beta = 0$ and at the limiting points A, B $\alpha = +\infty$ and $\alpha = -\infty$ respectively.

Putting $\zeta = \alpha + i\beta$ and $z = x + iy$ we have from (1)

$$\zeta = \log \frac{z + ia}{z - ia}. \tag{4}$$

Solving for z, we get

$$z = ia \frac{1 + e^{-\zeta}}{1 - e^{-\zeta}} = ae^{i(4k+1)\pi/2} \frac{1 + e^{-\zeta}}{1 - e^{-\zeta}}$$

where k is any positive integer,

$= re^{i\theta}$, r, θ being the usual polar coordinates of the point (x, y). Whence,

$$\log r = \log a + 2 \sum_{n=1,3,\ldots}^{\infty} \frac{e^{-n\alpha}}{n} \cos n\beta, \tag{5}$$

and

$$\theta = \frac{\pi}{2} - 2\sum_{n=1,3,\ldots}^{\infty} \frac{e^{-n\alpha}}{n} \sin n\beta, \quad (\alpha > 0)$$

(the principal values being taken).

The stresses in terms of the stress function χ are given by (Jeffery 1921, p. 269)

$$a\widehat{\alpha\alpha} = \left\{(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\beta^2} - \sinh\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \cosh\alpha\right\}(h\chi),$$

$$a\widehat{\beta\beta} = \left\{(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\alpha^2} - \sinh\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \cos\beta\right\}(h\chi),$$

$$a\widehat{\alpha\beta} = -(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\alpha\partial\beta}(h\chi), \tag{6}$$

so that we have

$$a(\widehat{\alpha\alpha} - \widehat{\beta\beta}) = (\cosh\alpha - \cos\beta)\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\alpha^2} + 1\right)(h\chi). \tag{7}$$

The necessary and sufficient conditions for a boundary α = constant to be free from stress are (Jeffery 1921 p. 282) that on the boundary

$$\frac{\partial}{\partial\alpha}(h\chi) = \text{constant} = \rho \text{ (say)} \tag{8}$$

and

$$h\chi = \rho\tanh a + \sigma(\cosh\alpha\cos\beta - 1) + \tau\sin\beta, \tag{9}$$

where ρ, σ, τ are Michell's three constants of the boundary.

3. *Infinite plate containing two unstressed equal circular holes under the action of a couple of moment M midway between them.*

Let the holes be defined by $\alpha = c(>0)$ and $\alpha = -c$. The origin of coordinates being taken at the point of application of the couple. Now for a couple of moment M applied at the origin in the positive sense

$$\chi_0 = -\frac{M}{2\pi}\theta,$$

so that, omitting the constant term in χ_0, since its omission does not effect the stresses we have

$$h\chi_0 = \frac{M}{\pi a}(\cosh\alpha - \cos\beta)\sum_{n=1,3,\ldots}^{\infty} \frac{e^{-n\alpha}}{n}\sin n\beta \tag{10}$$

and this has different expansions on different sides of the line $\alpha = 0$.

Thus for $\alpha > 0$, we have

$$h\chi_0 = \frac{M}{2\pi a}\left\{\sum_{n=1,3,\ldots}^{\infty} \frac{1}{n}\left[e^{-(n-1)\alpha} + e^{-(n+1)\alpha}\right] - \sum_{n=2,4,\ldots}^{\infty}\left[\frac{e^{-(n-1)\alpha}}{n-1} + \frac{e^{-(n+1)\alpha}}{n+1}\right]\right\}\sin n\beta \tag{11}$$

and for $\alpha < 0$.

$$h\chi_0 = \frac{M}{2\pi a}\left\{\sum_{n=1,3,\ldots}^{\infty} \frac{1}{n}\left[e^{(n-1)\alpha} + e^{(n+1)\alpha}\right] - \sum_{n=2,4,\ldots}^{\infty}\left[\frac{e^{(n-1)\alpha}}{n-1} + \frac{e^{(n+1)\alpha}}{n+1}\right]\right\}\sin n\beta. \tag{12}$$

For the complete stress function we have to add to χ_0 a stress function χ_1 which will give no stress at infinity and is such that the complete stress function $(\chi_0+\chi_1)$ will give no stress over the boundaries $\alpha = \pm c$.

We assume,

$$h\chi_1 = A_1 \sinh 2\alpha \sin\beta + \sum_{n=2}^{\infty}\{A_n \sinh(n+1)\alpha + B_n \sinh(n-1)\alpha\}\sin n\beta, \tag{13}$$

so that the stresses calculated from χ_1 at infinity vanish.

Let ρ, σ, τ be the Michell's constants for the boundary $\alpha = c$. The complete stress function $\chi = \chi_0+\chi_1$ given by (11) and (13) satisfies the boundary conditions (8) and (9) for the boundary $\alpha = c$, so we have $\rho = \sigma = 0$ and

$$\frac{M}{\pi a}\cosh c e^{-c} + A_1 \sinh 2c = \tau,$$

$$2A_1 \cosh 2c - \frac{M}{\pi a} e^{-2c} = 0, \tag{14}$$

and for n even ≥ 2

$$A_n \sinh(n+1)c + B_n \sinh(n-1)c = \frac{M}{2\pi a}\left\{\frac{e^{-(n-1)c}}{n-1}+\frac{e^{-(n+1)c}}{n+1}\right\},$$

$$(n+1)A_n \cosh(n+1)c + B_n(n-1)\cosh(n-1)c = -\frac{M}{2\pi a}\{e^{-(n-1)c}+e^{-(n+1)c}\}, \tag{15}$$

for n odd ≥ 3.

$$A_n \sinh(n+1)c + B_n \sinh(n-1)c = -\frac{M}{2\pi a}\frac{1}{n}\{e^{-(n-1)c}+e^{-(n+1)c}\},$$

$$(n+1)A_n \cosh(n+1)c + B_n(n-1)\cosh(n-1)c = \frac{M}{2\pi a}\left\{\frac{n-1}{n}e^{-(n-1)c}+\frac{n+1}{n}e^{-(n+1)c}\right\}. \tag{16}$$

For the boundary $\alpha = -c$, the complete stress function $\chi = \chi_0+\chi_1$ given by (12) and (13) satisfies the boundary conditions (8) and (9) where ρ, σ, τ are now replaced by three other constants ρ', σ', τ'. In this case we get the same system of equations as (14), (15) and (16) with the difference that in those equations the signs of the coefficients $A_1, A_2 \ldots$, $B_1, B_2 \ldots$ are changed and in (14) τ is replaced by τ' and $\rho' = \sigma' = 0$.

Solving these equations, we get

$$A_1 = \frac{M}{2\pi a} e^{-2c}\operatorname{sech} 2c, \tag{17}$$

and for n even ≥ 2

$$A_n\{\sinh 2nc - n\sinh 2c\} = -\frac{M}{2\pi a}\frac{1}{n+1}\{n+1+ne^{-2c}-e^{-2nc}\},$$

$$B_n\{\sinh 2nc - n\sinh 2c\} = \frac{M}{2\pi a}\frac{1}{n-1}\{n-1+ne^{2c}+e^{-2nc}\}, \tag{18}$$

for n odd ≥ 3

$$A_n\{\sinh 2nc - n \sinh 2c\} = \frac{M}{2\pi a}\frac{1}{n}\{n-1+ne^{-2c}-e^{-2nc}\},$$

$$B_n\{\sinh 2nc - n \sinh 2c\} = -\frac{M}{2\pi a}\frac{1}{n}\{n+1+ne^{2c}+e^{-2nc}\}. \tag{19}$$

On the hole $\alpha = c$, $\widehat{\alpha\alpha} = 0$ so that $\widehat{\beta\beta}$ can be very easily calculated from (7). We have,

$$\frac{\pi a^2}{2M}\widehat{\beta\beta}_c = (\cosh c - \cos\beta)\left[\operatorname{sech} 2c \sin\beta + \sum_{n=3,5,\ldots}^{\infty} R_n \sin n\beta - \sum_{n=2,4,\ldots}^{\infty} S_n \sin n\beta\right], \tag{20}$$

where,

$$[R_n + e^{-(n-1)c}](\sinh 2nc - n\sinh 2c) = [(n+1)+ne^{2c}+e^{-2nc}]\sinh(n-1)c,$$

and

$$\left[\frac{n-1}{n}S_n + e^{-(n-1)c}\right](\sinh 2nc - n\sinh 2c) = (n-1+ne^{2c}+e^{-2nc})\sinh(n-1)c.$$

Except for large values of c the series in (20) are slowly convergent. To obtain the expressions in the form of a more rapidly convergent series, we put

$$R_n = 2e^{-nc}(n\cosh c - \sinh c) + N_n - M_n,$$

and

$$S_n = 2ne^{-nc}\cosh c + N_n, \tag{21}$$

where,

$$N_n(\sinh 2nc - n\sinh 2c) = ne^{-nc}\cosh c[2n\sinh 2c - 1 + e^{-2nc}],$$

and

$$M_n(\sinh 2nc - n\sinh 2c) = e^{-nc}\sinh c[2n\sinh 2c + 1 + e^{-2nc}].$$

Hence,

$$\frac{\pi a^2}{2M}\widehat{\beta\beta}_c = (\cosh c - \cos\beta)\Bigg[\operatorname{sech} 2c \sin\beta - 2\sum_{n=2}^{\infty}(-1)^n ne^{-nc}\cosh c \sin n\beta$$

$$-2\sum_{n=3,5,\ldots}^{\infty} e^{-nc}\sinh c \sin n\beta - \left\{\sum_{n=2}^{\infty}(-1)^n N_n \sin n\beta + \sum_{n=3,5,\ldots}^{\infty} M_n \sin n\beta\right\}\Bigg]. \tag{22}$$

Noting that,

$$2(\cosh c - \cos\beta)\left[\sum_{n=2}^{\infty}(-1)^n ne^{-nc}\cosh c \sin n\beta + \sum_{n=3,5,\ldots}^{\infty} e^{-nc}\sinh c \sin n\beta\right]$$

$$= \frac{\sinh 2c \sin\beta\cos\beta}{(\cosh c + \cos\beta)^2} + 2e^{-2c}(\cosh c - \cos\beta)\sin\beta.$$

We have,

$$\frac{\pi a^2}{2M}\widehat{\beta\beta}_c = -\frac{\sinh 2c\sin\beta\cos\beta}{(\cosh c + \cos\beta)^2} - (\cosh c - \cos\beta)\Bigg[e^{-4c}\operatorname{sech} 2c \sin\beta$$

$$+\sum_{n=2}^{\infty}(-1)^n N_n \sin n\beta + \sum_{n=3,5,\ldots}^{\infty} M_n \sin n\beta\Bigg]. \tag{23}$$

The table I gives the values of $\widehat{\beta\beta}$ on the boundary of the hole $\alpha = c$ for positive values of β when $c = \cdot 8$ for which the shortest distance between the boundary of the hole and the axis of the couple is approximately one third the radius of the hole. The

ratio of the distance between the centres of the holes to the diameter of a hole being $\cosh c(=1\cdot34)$. From the same table the value of $\widehat{\beta\beta}$ for negative values of β are obtained by changing the sign of $\widehat{\beta\beta}$. For positive values of β, the maximum tension is found after calculation to be $6\cdot013(2M/\pi a^2)$ at the point $\beta = 151°$ approximately and the maximum compression in this case is $-\cdot748(2M/\pi a^2)$ at the point $\beta = 60°$ approximately.

TABLE I

$c = \cdot8$

β	$\frac{\pi a^2}{2M}\widehat{\beta\beta}$	β	$\frac{\pi a^2}{2M}\widehat{\beta\beta}$	β	$\frac{\pi a^2}{2M}\widehat{\beta\beta}$
0°	0·000	60°	−·7476	120°	2·4115
15°	−·2132	75°	−·5028	135°	4·334
30°	−·4387	90°	−·09702	150°	5·9732
45°	−·640	150°	·9012	165°	5·1486
				180°	0·000

4 *Infinite plate containing two unstressed equal circular holes, subjected to a centre of pressure radiating from a point midway between them.*

Let the holes be defined by $\alpha = c(>0)$ and $\alpha = -c$. The origin of coordinates being taken at the centre of pressure. Now for a centre of pressure radiating from the origin

$$\chi_0 = \log r,$$

so that omitting the constant term in χ_0, since its omission does not effect the stresses, we have, for $\alpha > 0$

$$h\chi_0 = \frac{1}{2a}(\cosh\alpha-\cos\beta)\log\frac{\cosh\alpha+\cos\beta}{\cosh\alpha-\cos\beta} = \frac{2}{a}(\cosh\alpha-\cos\beta)\sum_{n=1,3,..}^{\infty}\frac{e^{-n\alpha}}{n}\cos n\beta. \quad (24)$$

This has different expansions on different sides of the line $\alpha = 0$.

Thus for $\alpha > 0$

$$h\chi_0 = -\frac{e^{-\alpha}}{a}+\frac{1}{a}\sum_{n=1,3,..}^{\infty}\frac{1}{n}\{e^{-(n-1)\alpha}+e^{-(n+1)\alpha}\}\cos n\beta$$
$$-\frac{1}{a}\sum_{n=2,4,}^{\infty}\left\{\frac{e^{-(n-1)\alpha}}{n-1}+\frac{e^{-(n+1)\alpha}}{n+1}\right\}\cos n\beta, \quad (25)$$

and for $\alpha < 0$

$$h\chi_0 = -\frac{e^{\alpha}}{a}+\frac{1}{a}\sum_{n=1,3,..}^{\infty}\frac{1}{n}\{e^{(n-1)\alpha}+e^{(n+1)\alpha}\}\cos n\beta-\frac{1}{a}\sum_{n=2,4,}^{\infty}\left\{\frac{e^{(n-1)\alpha}}{n-1}+\frac{e^{(n+1)\alpha}}{n+1}\right\}\cos n\beta. \quad (26)$$

For a complete stress function we have to add to χ_0 a stress function χ_1 which will give no stress at infinity and is such that the complete stress function $(\chi_0+\chi_1)$ will give no stress over the boundaries $\alpha = \pm c$. We assume

$$h\chi_1 = B_0\alpha(\cosh\alpha-\cos\beta)+A_1(\cosh 2\alpha-1)\cos\beta$$
$$+\sum_{2}^{\infty}\{A_n[\cosh(n+1)\alpha-\cosh(n-1)\alpha]$$
$$+B_n[(n-1)\sinh(n+1)\alpha-(n+1)\sinh(n-1)\alpha]\}\cos n\beta. \tag{27}$$

so that the stresses due to χ_1 at infinity vanish.

Let ρ, σ, τ be Michell's three constants of the boundary $\alpha = c$. The complete stress function $(\chi_0+\chi_1)$ given by (25) and (27) satisfies the boundary conditions (8) and (9) for the boundary $\alpha = c$. So, we have $\tau = 0$ and

$$\frac{e^{-c}}{a}+B_0(c\sinh c+\cosh c) = \rho, \tag{28}$$

$$-\frac{e^{-c}}{a}+B_0 c\cosh c = \rho\tanh c-\sigma,$$

$$\frac{1}{a}(1+e^{-2c})-B_0c+A_1(\cosh 2c-1) = \sigma\cosh c,$$

and

$$2A_1\sinh 2c-B_0-\frac{2}{a}e^{-2c} = 0,$$

and for n even ≥ 2

$$A_n[\cosh(n+1)c-\cosh(n-1)c]$$
$$+B_n[(n-1)\sinh(n+1)c-(n+1)\sinh(n-1)c]$$
$$= \frac{1}{a}\left\{\frac{e^{-(n-1)c}}{n-1}+\frac{e^{-(n+1)c}}{n+1}\right\},$$

$$A_n[(n+1)\sinh(n+1)c-(n-1)\sinh(n-1)c]$$
$$+B_n(n^2-1)[\cosh(n+1)c-\cosh(n-1)c] = -\frac{1}{a}\{e^{-(n-1)c}+e^{-(n+1)c}\}, \tag{29}$$

for n odd ≥ 3

$$A_n[\cosh(n+1)c-\cosh(n-1)c]$$
$$+B_n[(n-1)\sinh(n+1)c-(n+1)\sinh(n-1)c] = -\frac{1}{an}\{e^{-(n-1)c}+e^{-(n+1)c}\},$$

$$A_n[(n+1)\sinh(n+1)c-(n-1)\sinh(n-1)c]$$
$$+B_n(n^2-1)[\cosh(n+1)c-\cosh(n-1)c]$$
$$= \frac{1}{a}\left\{\frac{n-1}{n}e^{-(n-1)c}+\frac{n+1}{n}e^{-(n+1)c}\right\}. \tag{30}$$

For the boundary $\alpha = -c$ the complete stress function $\chi = (\chi_0+\chi_1)$ given by (26) and (27) satisfies the boundary conditions (8) and (9) where ρ, σ, τ are replaced by three other constants ρ', σ', τ'. In this case we get the same systems of equations (28), (29) and (30) with the difference that in these equations the signs of the coefficients B_0, B_n are changed and in (28) ρ and σ are replaced by $-\rho$ and σ', τ' being zero.

Solving these equations, we get

$$A_1 = \frac{e^{-2c}(1+\sinh 2c)}{a\cosh 2c(\cosh 2c-1)}, \qquad B_0 = \frac{4e^{-c}\sinh c}{a\cosh 2c(\cosh 2c-1)}, \tag{31}$$

and for n even ≥ 2

$$A_n(\sinh^2 nc - n^2\sinh^2 c) = \frac{1}{a}\{n\sinh c\cosh c - e^{-nc}\sinh nc\},$$

$$B_n(\sinh^2 nc - n^2\sinh^2 c) = -\frac{n}{a(n^2-1)}\{n\sinh c\cosh c + \cosh^2 c - e^{-nc}\cosh nc\}, \tag{32}$$

for n odd ≥ 3

$$A_n(\sinh^2 nc - n^2\sinh^2 c) = \frac{1}{a}\{n\sinh c\cosh c - \sinh^2 c - e^{-nc}\sinh nc\},$$

$$B_n(\sinh^2 nc - n^2\sinh^2 c) = -\frac{1}{an}\{n\sinh c\cosh c + e^{-nc}\sinh nc\}. \tag{33}$$

On the hole $\alpha = c$, $\widehat{\alpha\alpha} = 0$, so that $\widehat{\beta\beta}$ can be very easily calculated from (7). Then

$$\frac{a^2}{4}\widehat{\beta\beta}_c = (\cosh c - \cos\beta)\Big[e^{-c}\operatorname{sech} 2c + \tfrac{1}{2}\operatorname{cosech}^2 c\cos\beta + \sum_{n=3,5,}^{\infty} R_n\cos n\beta + \sum_{n=2,4,\,.}^{\infty} S_n\cos n\beta\Big], \tag{34}$$

where, $[R_n + e^{-(n-1)c}](\sinh^2 nc - n^2\sinh^2 c)$

$$= n[n\sinh c\cosh c - \sinh^2 c - e^{-nc}\sinh nc]\cosh(n-1)c - (n+1)[n\sinh c\cosh c + e^{-nc}\sinh nc]\sinh(n-1)c, \tag{35}$$

and $\left[\frac{n-1}{n}S_n - e^{-(n-1)c}\right](\sinh^2 nc - n^2\sinh^2 c)$

$$= (n-1)[n\sinh c\cosh c - e^{-nc}\sinh nc]\cosh(n-1)c - n[n\sinh c\cosh c + \cosh^2 c - e^{-nc}\cosh nc]\sinh(n-1)c. \tag{36}$$

Except for large values of c the series in (34) are slowly convergent. To make them rapidly convergent we put

$$R_n = -2(n+1)e^{-nc}\cosh c - N_n + M_n,$$

and

$$S_n = -2ne^{-nc}\cosh c - N_n \tag{37}$$

where,

$$N_n(\sinh^2 nc - n^2\sinh^2 c) = ne^{-nc}[n\sinh c(n\sinh 2c - 1) + e^{-nc}\cosh c\sinh nc]$$

and

$$M_n(\sinh^2 nc - n^2\sinh^2 c) = e^{-nc}[2n^2\sinh^2 c + e^{-nc}\sinh nc]\sinh c.$$

So that

$$\frac{a^2}{4}\widehat{\beta\beta}_c = (\cosh c - \cos\beta)\{e^{-c}\operatorname{sech} 2c + \tfrac{1}{2}\operatorname{cosech}^2 c\cos\beta\}$$

$$-2(\cosh c - \cos\beta)\Big[\sum_2^{\infty} ne^{-nc}\cosh c\cos n\beta + \sum_{n=3,5,\,.}^{\infty} e^{-nc}\cosh c\cos n\beta\Big]$$

$$-(\cosh c - \cos\beta)\Big[\sum_2^{\infty} N_n\cos n\beta - \sum_{n=3,5,\,.}^{\infty} M_n\cos n\beta\Big]. \tag{38}$$

Noting that

$$2(\cosh c-\cos\beta)\sum_{1}^{\infty} ne^{-nc}\cosh c\cos n\beta = \frac{\cosh c\,(\cosh c\cos\beta-1)}{\cosh c-\cos\beta}$$

and

$$2(\cosh c-\cos\beta)\sum_{n=1,3,\ldots}^{\infty} e^{-nc}\cosh c\cos n\beta = \frac{\cosh c\sinh c\cos\beta}{\cosh c+\cos\beta}, \tag{39}$$

we have

$$\frac{a^2}{4}\widehat{\beta\beta}_c = (\cosh c-\cos\beta)\left[e^{-c}\operatorname{sech} 2c+\frac{e^{-4c}+2\sinh 2c}{\cosh 2c-1}\cos\beta\right]$$

$$-\frac{\cosh c(\cosh c\cos\beta-1)}{\cosh c-\cos\beta}-\frac{\cosh c\sinh c}{\cosh c+\cos\beta}\cos\beta$$

$$-(\cosh c-\cos\beta)\left[\sum_{2}^{\infty} N_n\cos n\beta-\sum_{n=3,5,\ldots}^{\infty} M_n\cos n\beta\right]. \tag{40}$$

The stresses are symmetrical about the y-axis. Hence, in the table II the values of $\widehat{\beta\beta}$ on the boundary $\alpha = c$ for values of β between 0 and π are given when $c = 1$, *i.e.* when the ratio of the distance between the centres of the holes to the diameter of a hole is $\cosh c(= 1\cdot 54)$. From the table it will be found that the maximum compressions are at the extremities of the diameter of the hole passing through the 'centre of pressure' and it is more marked at the point nearest the 'centre of pressure'. The points of zero stresses are found to be approximately at $\beta = 29°51'$ and at $\beta = 122°20'$ with the point of maximum tension $(4/a^2)\times 2\cdot 132$ at $\beta = 79°$ approximately.

TABLE II

$c = 1$

β	$\frac{a^2}{4}\widehat{\beta\beta}$	β	$\frac{a^2}{4}\widehat{\beta\beta}$	β	$\frac{a^2}{4}\widehat{\beta\beta}$
0°	−1·0985	60°	1·7034	120°	·1973
15°	−·7735	75°	1·8873	135°	−1·0574
30°	·0089	90°	1·987	150°	−2·0961
45°	·9308	105°	1·4637	165°	−2·69
				180°	−2·86

In conclusion I thank Dr. B Sen for his kind help in the preparation of this paper.

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NOTE ON CERTAIN INTEGRALS AND SERIES INVOLVING TSCHEBYSCHEFF'S FUNCTIONS

BY

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INTRODUCTION

The present paper deals with the two kinds of Tschebyscheff's functions $T_n(z)$ and $V_n(z)$ and is simply a continuation of our two other papers (Bagchi and Chakrabarti, 1950), entitled,

(*i*) "Note on Tschebyscheff's function $T_n(z)$ and its associated equations",

and (*ii*) "Some further relations connected with Tschebyscheff's function $T_n(z)$". The subject-matter, discussed in this paper, may, broadly speaking, be classified under three heads, *viz.*,

(*a*) the establishment of certain definite integrals equivalent to $T_n(z)$ and $V_n(z)$;

(*b*) the scrutinisation of their asymptotic values,

and (*c*) the *conditional* summation of certain infinite series involving them.

We are not aware whether these results have been discussed heretofore by any previous writer.

1. Starting with the proved result: (Bagchi and Chakrabarti, 1950, p. 46)

$$\frac{T_n(z)}{(z^2-1)^{\frac{1}{2}}} = \lambda \int_0^{\pi} \cos \frac{\phi}{2} (z + \sqrt{z^2-1} \cos \phi)^{n-\frac{1}{2}} d\phi, \tag{1}$$

where

$$\lambda = \frac{2^{n+\frac{1}{2}}(n\,!)^2}{\pi(2n)\,!}, \tag{2}$$

and applying the transformation

$$\cos t = \frac{z \cos \phi + \sqrt{z^2-1}}{z + \sqrt{z^2-1} \cos \phi},$$

we are led to the relation

$$\frac{T_n(z)}{(z^2-1)^{\frac{1}{2}}} = \frac{\lambda}{(z + \sqrt{z^2-1})^{\frac{1}{2}}} \int_0^{\pi} \frac{\sin \frac{t}{2}\, dt}{(z + \sqrt{z^2-1} \cos t)^{n+1}}. \tag{3}$$

Manifestly this result, which is deducible alternatively from (1) by the substitution:

$$z + \sqrt{z^2-1} \cos \phi = \frac{1}{z + \sqrt{z^2-1} \cos \theta},$$

may be regarded as the analogue of Laplace's second integral formula for $P_n(z)$.

To find the corresponding integral formula for $V_n(z)$, we utilise the known result (Bagchi and Chakrabarti, 1950, p 40)

$$V_n(z) = \frac{\prod(n)}{4^n\left\{\prod\left(\frac{n-1}{2}\right)\right\}^2}\cdot\frac{1}{z^n}\int_0^1 v^{\frac{n-1}{2}}(1-v)^{\frac{n-1}{2}}(1-vz^{-2})^{-\frac{n}{2}}dv,\ (|z|>1) \tag{4}$$

and change the variable according to the scheme

$$v = \frac{\cosh\psi-1}{\cosh\psi+1}.$$

If we now write

$$z = \mu+\sqrt{\mu^2-1}, \quad \text{so that} \quad \mu = \frac{z+z^{-1}}{2},$$

the formula (4) can be transformed into

$$z^{\frac{n}{2}}V_n(z) = \frac{1}{4^n}\cdot\frac{\prod(n)}{\left\{\prod\left(\frac{n-1}{2}\right)\right\}^2}\cdot\int_0^\infty\frac{\tanh^n\frac{\psi}{2}\operatorname{sech}\frac{\psi}{2}}{(\mu+\sqrt{\mu^2-1}\cosh\psi)^{\frac{n}{2}}}d\psi. \tag{5}$$

Plainly (5) resembles Laplace's integral formula for $Q_n(z)$.

2. We shall now scrutinise the values of $T_n(z)$ and $V_n(z)$, where n is a very large positive integer.

Recalling Stirling's formula:

$$n! = e^{-n}.n^n.(2\pi n)^{\frac{1}{2}}, \quad \text{(when } n \text{ is very large)}$$

we readily perceive from (2) and (3) that, for very large values of n,

$$\left|\frac{T_n(z)}{(z^2-1)^{\frac{1}{4}}}\right| \leq \frac{\sqrt{(n\pi)}}{2^{n-\frac{1}{2}}}\cdot|z\pm\sqrt{z^2-1}|^{n-\frac{1}{2}}. \tag{6}$$ †

In the particular case $-1<z<1$, (6) plainly reduces to

$$|T_n(z)| \leq \frac{\sqrt{(n\pi)}}{2^{n-\frac{1}{2}}}. \tag{7}$$

Next, the formula (Bagchi and Chakrabarti, 1950, p. 40)

$$V_n(z) = \frac{1}{2^{2n-1}}\cdot\frac{\prod(n)}{\left\{\prod\left(\frac{n-1}{2}\right)\right\}^2}\cdot\int_0^{\frac{\pi}{2}}\frac{\sin^n\theta\cos^n\theta d\theta}{(z^2-\sin^2\theta)^{\frac{1}{2}n}}, \quad (|z|>1)$$

gives rise to the inequality

$$|V_n(z)| \leq \frac{1}{2^{2n}}\cdot\frac{\prod(n)}{\left\{\prod\left(\frac{n-1}{2}\right)\right\}^2}\cdot\frac{\pi}{(|z|^2-1)^{\frac{1}{2}n}},$$

† For obvious reasons, the relative magnitudes of the two positive quantities

(i) $a = |z+\sqrt{z^2-1}|$ and (ii) $b = |z-\sqrt{z^2-1}|$

depend upon the initial values of z (real or imaginary). Thus, for instance, when z is real and >1, $a>b$; but when z is real and <-1, $a<b$. In the above context, that particular sign $+$ or $-$, which corresponds to the greater of the moduli (i), (ii), is to be chosen on the R. S. of (6).

which, taken in conjunction with Stirling's formula, ultimately leads to

$$|V_n(z)| \leq \frac{\sqrt{(n\pi)}}{2^{n+\frac{1}{2}}(|z|^2-1)^{\frac{1}{2}n}}, \quad (|z|>1) \tag{8}$$

when n is very large.

3. By way of illustration of the utility of the *asymptotic* values of $T_n(z)$ and $V_n(z)$, as given by (6), (7) and (8), we now proceed to sum (when possible) each of the three infinite series:

$$A = \sum_{r=1}^{\infty} 4^{r-1} T_r(x) V_r(y), \tag{9}$$

$$B = \sum_{r=1}^{\infty} 4^{r-1} T_r(x) T_r(y), \tag{10}$$

and
$$C = \sum_{r=1}^{\infty} 4^{r-1} V_r(x) V_r(y). \tag{11}$$

At the very outset we utilise the proved result (Bagchi and Chakrabarti, 1950, p. 39),

$$\sum_{r=1}^{r=n} 4^{r-1}\psi_r(x)\phi_r(y) = \frac{1}{4(y-x)}[\psi_1(x)\phi_0(y)-\psi_0(x)\phi_1(y)] - \bar{R}_n(x, y), \tag{12}$$

where $\psi_n(z)$ and $\phi_n(z)$ are any two particular (analytic) solutions of the functional equation

$$u_{n+1}(z) - zu_n(z) + \tfrac{1}{4}u_{n-1}(z) = 0, \tag{13}$$

and

$$\bar{R}_n(x, y) = \frac{4^{n-1}}{y-x}.[\psi_{n+1}(x)\phi_n(y) - \psi_n(x)\phi_{n+1}(y)].$$

Remarking that $T_n(z)$ and $V_n(z)$ are but special solutions of (13) and choosing the functions ψ_n and ϕ_n appropriately, we can readily derive three summation-formulae involving the two kinds of Tschebyscheff's functions. In fact, if A_n, B_n and C_n denote respectively the sums of the first n terms of the series (9), (10) and (11), the formulae in question may be presented in the following forms:

$$A_n = \frac{x - V_1(y)}{4(y-x)} - R_n^{(1)}(x, y), \tag{14}$$

$$B_n = -\tfrac{1}{4} - R_n^{(2)}(x, y), \tag{15}$$

and
$$C_n = \frac{V_1(x) - V_1(y)}{4(y-x)} - R_n^{(3)}(x, y), \tag{16}$$

it being implied that

$$R_n^{(1)}(x, y) = \frac{4^{n-1}}{y-x}.[T_{n+1}(x)V_n(y) - T_n(x)V_{n+1}(y)], \tag{17}$$

$$R_n^{(2)}(x, y) = \frac{4^{n-1}}{y-x}.[T_{n+1}(x)T_n(y) - T_n(x)T_{n+1}(y)], \tag{18}$$

and
$$R_n^{(3)}(x, y) = \frac{4^{n-1}}{y-x} . [V_{n+1}(x)V_n(y) - V_n(x)V_{n+1}(y)] . \tag{19}$$

There is not much difficulty in verifying, by means of (6), (7) and (8), that when $n \to \infty$,

$$\left.\begin{array}{lll} (a) & R_n^{(1)}(x, y) \to 0, \text{ if } |x \pm \sqrt{x^2-1}\,|^2 < |y|^2 - 1, \text{ and } (|y| > 1) & (20) \\ (b) & R_n^{(2)}(x, y) \to 0, \text{ if } |x \pm \sqrt{x^2-1}\,|\,|y \pm \sqrt{y^2-1}\,| < 1, \text{ and } (|y| > 1) & (21) \end{array}\right\} (\dagger)$$

$$(c) \quad R_n^{(3)}(x, y) \to 0, \text{ if } (|x|^2 - 1)(|y|^2 - 1) > 1, \text{ and } (|x| > 1, |y| > 1). \tag{22}$$

It is scarcely necessary to add that, subject to the conditions of convergence (20), (21), (22), the three infinite series A, B, C, given by (9), (10), (11), converge respectively to the sums

$$\frac{x - V_1(y)}{4(y-x)}, \quad -\frac{1}{4}, \quad \frac{V_1(x) - V_1(y)}{4(y-x)}.$$

One may attempt a geometrical interpretation (with reference to the Argand plane) of the three algebraic conditions of convergency (20), (21) and (22) of the three series A, B, C.

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† The ambiguity in the algebraic sign $\pm$, occurring in the L. S. of (20) and (21) is to be removed in accordance with the principle, set forth in the foot-note (†) of Art. 2.

TORSION AND FLEXURE OF A BEAM WHOSE CROSS-SECTION IS A SECTOR OF A CURVE

By

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1. In this paper the torsion-flexure problem for the sector of any curve is solved by an extension of the method used in a previous paper (Mitra, 1950). By two successive conformal transformations the area of the sector is represented on the upper half of the unit circle. By the application of Schwarz's principle of reflection, the problem is reduced to the determination of analytic functions within the unit circle, whose imaginary parts take up prescribed values on the circumference. The functions are then determined with the help of Schwarz's formula.

Torsion Problem

2. Let the origin be taken at the point of intersection of the two bounding straight lines of the cross-section $\theta = 0$ and $\theta = 2\delta$, 2δ being the angle of the sector. Let

$$z = t^m, \quad m = 2\delta/\pi \tag{2.1}$$

be the formula transforming conformally the area of the sector on an area bounded partly by a portion of the real axis of the t-plane. Let

$$t = \omega(\zeta) \tag{2.2}$$

be the mapping function which represents conformally the transformed area into the upper half of the unit circle in the ζ-plane, the straight boundary of the area in the t-plane being transformed into the bounding diameter of the semi-circle. Let α, β denote the upper and lower semi-circumferences of the circumference of the unit circle. The imaginary part of the complex torsion function $F_0(\zeta)$ has the value $\frac{1}{2}[\omega(\zeta)\bar{\omega}(\bar{\zeta})]^m$ on the boundary. If we choose the function

$$G_0(\zeta) = F_0(\zeta) - \tfrac{1}{2}(i + \tan 2\delta)[\omega(\zeta)]^{2m} \tag{2.3}$$

($2\delta \neq \pi/2,\ 3\pi/2$), it is easily seen that the imaginary part of $G_0(\zeta)$ vanishes on the real axis and takes up the value

$$= \tfrac{1}{2}[\omega(\zeta)\bar{\omega}(1/\zeta)]^m - \tfrac{1}{4}[\{\omega(\zeta)\}^{2m} + \{\bar{\omega}(1/\zeta)\}^{2m}]$$
$$+ \tfrac{1}{4}i \tan 2\delta[\{\omega(\zeta)\}^{2m} - \{\bar{\omega}(1/\zeta)\}^{2m}] = X_1 \text{ (say)}. \tag{2.4}$$

on the semi-circumference α. If we continue the function $G_0(\zeta)$ to the lower half of the unit circle by Schwarz's principle of reflection, the imaginary part of the function takes up the value $-X_1$ at the point $\bar{\zeta}$ on β. Applying Schwarz's formula and putting $\bar{\zeta} = 1/\zeta$, we get

$$G_0(\zeta) = \frac{1}{2\pi}\left[\int_\alpha \{\omega(t)\bar{\omega}(1/t)\}^m \frac{dt}{t-\zeta} - \int_\beta \{\omega(1/t)\bar{\omega}(t)\}^m \frac{dt}{t-\zeta}\right]$$

$$-\frac{1}{4\pi}\left[\int_\alpha\{[\omega(t)]^{2m}+[\bar{\omega}(1/t)]^{2m}\}\frac{dt}{t-\zeta}-\int_\beta\{[\omega(1/t)]^{2m}+[\bar{\omega}(t)]^{2m}\}\frac{dt}{t-\zeta}\right]$$

$$-\frac{\tan 2\delta}{4\pi i}\left[\int_\alpha\{[\omega(t)]^{2m}-[\bar{\omega}(1/t)]^{2m}\}\frac{dt}{t-\zeta}-\int_\beta\{[\omega(1/t)]^{2m}-[\bar{\omega}(t)]^{2m}\}\frac{dt}{t-\zeta}\right]. \quad (2.5)$$

When $\delta = \pi/4$, we choose

$$G_0(\zeta) = F_0(\zeta)-\tfrac{1}{2}i\omega(\zeta)+\frac{\omega(\zeta)}{\pi}\log\zeta \quad (2.6)$$

and we get

$$G_0(\zeta) = \frac{1}{2\pi}\left[\int_\alpha\{\omega(t)\bar{\omega}(1/t)\}^{\frac{1}{2}}\frac{dt}{t-\zeta}-\int_\beta\{\omega(1/t)\bar{\omega}(t)\}^{\frac{1}{2}}\frac{dt}{t-\zeta}\right]$$

$$-\frac{1}{4\pi}\left[\int_\alpha\{\omega(t)+\bar{\omega}(1/t)\}\frac{dt}{t-\zeta}-\int_\beta\{\omega(1/t)+\bar{\omega}(t)\}\frac{dt}{t-\zeta}\right]$$

$$-\frac{i}{2\pi^2}\left[\int_\alpha\log t\{\omega(t)+\bar{\omega}(1/t)\}\frac{dt}{t-\zeta}+\int_\beta\log t\{\omega(1/t)+\bar{\omega}(t)\}\frac{dt}{t-\zeta}\right]. \quad (2.7)$$

Flexure Problem

3. The solution of the flexure problem lies in determining two analytic functions whose imaginary parts ψ_1 and ψ_2 take up the values (Ghosh, 1948, p 77)

$$\psi_1 = [(1+\sigma)a^2-\sigma b^2]y-(1+\sigma)axy+\sigma by^2-\tfrac{1}{3}(1+2\sigma)y^3+(1+\sigma)\mathbf{I}\int_A^P(z-a)\bar{z}dz, \quad (3.1)$$

$$\psi_2 = [\sigma a^2-(1+\sigma)b^2]x-\sigma ax^2+(1+\sigma)bxy+\tfrac{1}{3}\sigma x^3$$
$$-(1+\sigma)b\mathbf{I}\int_A^P\bar{z}dz+\tfrac{1}{4}(1+\sigma)\mathbf{R}\int_A^P(z-\bar{z})^2dz \quad (3.2)$$

on the boundary, where a, b denote the co-ordinates of the $C.G.$ of the area of the cross-section. Let $F_1(\zeta)$ and $F_2(\zeta)$ be the complex flexure functions whose imaginary parts satisfy conditions (3.1) and (3.2) on the boundary of the cross-section. If we choose

$$G_1(\zeta) = F_1(\zeta)-\{(1+\sigma)a^2-\sigma b^2\}[\omega(\zeta)]^m+\tfrac{1}{2}(1+\sigma)a[\omega(\zeta)]^{2m}$$
$$-\frac{1}{2}\frac{\sigma b\sin 4\delta}{(1+\cos 4\delta)}[\omega(\zeta)]^{2m}-\frac{(1+2\sigma)(\sin 6\delta-3\sin 2\delta)}{12\sin 6\delta}[\omega(\zeta)]^{3m}-\frac{(1+\sigma)\sin 2\delta}{3\sin 6\delta}[\omega(\zeta)]^{3m}, \quad (3.3)$$

$$G_2(\zeta) = F_2(\zeta)-i\{\sigma a^2-(1+\sigma)b^2\}[\omega(\zeta)]^m+\tfrac{1}{2}i\sigma a(2-i\tan 2\delta)[\omega(\zeta)]^{2m}-\tfrac{1}{2}(1+\sigma)b[\omega(\zeta)]^{2m}$$
$$-\frac{1}{3}\sigma\left[i+\frac{3(\cos 2\delta-\cos 6\delta)}{4\sin 6\delta}\right][\omega(\zeta)]^{3m}+\frac{(1+\sigma)\sin^2 2\delta\cos 2\delta}{3\sin 6\delta}[\omega(\zeta)]^{3m}, \quad (3.4)$$

$G_1(\zeta)$ and $G_2(\zeta)$ are analytic within the upper semi-circle and their imaginary parts will respectively take the values (3.5) and (3.6) on the boundary of the semi-circle

$$-\tfrac{1}{4}\sigma b[\{\omega(\zeta)\}^{2m}+\{\bar{\omega}(\bar{\zeta})\}^{2m}-2\{\omega(\zeta)\bar{\omega}(\bar{\zeta})\}^{m}]-\frac{1+2\sigma}{8i}\{\omega(\zeta)\bar{\omega}(\bar{\zeta})\}^{m}[\{\omega(\zeta)\}^{m}-\{\bar{\omega}(\bar{\zeta})\}^{m}]$$

$$+\tfrac{1}{4}\sigma bi\frac{(1-\cos 4\delta)}{\sin 4\delta}[\{\omega(\zeta)\}^{2m}-\{\bar{\omega}(\bar{\zeta})\}^{2m}]+\frac{(2\sigma-1)\sin 2\delta}{24i\sin 6\delta}[\{\omega(\zeta)\}^{3m}-\{\bar{\omega}(\bar{\zeta})\}^{3m}]$$

$$+(1+\sigma)\frac{F(\zeta)-\bar{F}(\bar{\zeta})}{2i}=X(\text{say}) \quad (3.5)$$

$$\tfrac{1}{4}\sigma a\left[\{\omega(\zeta)\}^{2m}+\{\bar{\omega}(\bar{\zeta})\}^{2m}-2\{\omega(\zeta)\bar{\omega}(\bar{\zeta})\}^{m}+\tan 2\delta\frac{\{\omega(\zeta)\}^{2m}-\{\bar{\omega}(\bar{\zeta})\}^{2m}}{i}\right]$$

$$-\tfrac{1}{8}\sigma[\{\omega(\zeta)\}^{3m}+\{\bar{\omega}(\bar{\zeta})\}^{3m}-\{\omega(\zeta)\bar{\omega}(\bar{\zeta})\}^{m}\{[\omega(\zeta)]^{m}+[\bar{\omega}(\bar{\zeta})]^{m}\}]$$

$$+\frac{(1-2\sigma)\sin^2 2\delta\cos 2\delta}{6i\sin 6\delta}[\{\omega(\zeta)\}^{3m}-\{\bar{\omega}(\bar{\zeta})\}^{3m}]-(1+\sigma)b\frac{G(\zeta)-\bar{G}(\bar{\zeta})}{2i}$$

$$+\tfrac{1}{8}(1+\sigma)[H(\zeta)+\bar{H}(\bar{\zeta})]=Y(\text{say}) \quad (3.6)$$

where

$$F(\zeta)=\int_1^{\zeta}[\{\omega(\zeta)\}^{m}-a]\{\bar{\omega}(\bar{\zeta})\}^{m}d\{\omega(\zeta)\}^{m}, \quad (3.7)$$

$$G(\zeta)=\int_1^{\zeta}\{\bar{\omega}(\bar{\zeta})\}^{m}d\{\omega(\zeta)\}^{m}, \quad (3.8)$$

$$H(\zeta)=\int_1^{\zeta}[\{\omega(\zeta)\}^{m}-\{\bar{\omega}(\bar{\zeta})\}^{m}]^2d\{\omega(\zeta)\}^{m}. \quad (3.9)$$

It is readily seen from (3.5) and (3.6) that the imaginary parts of $G_1(\zeta)$ and $G_2(\zeta)$ vanish on the real axis. When $G_1(\zeta)$ and $G_2(\zeta)$ are continued analytically to the lower half of the unit circle, their imaginary parts take respectively the values $-X$, $-Y$ at the point $\bar{\zeta}$ on β. Putting $\bar{\zeta}=1/\zeta$ in the boundary values X and Y on the circumference of the unit circle and applying Schwarz's formula we get,

$$G_1(\zeta)=-\frac{\sigma b}{4\pi}\Big[\int_{\alpha}\{[\omega(t)]^{2m}+[\bar{\omega}(1/t)]^{2m}-2[\omega(t)\bar{\omega}(1/t)]^{m}\}\frac{dt}{t-\zeta}$$

$$-\int_{\beta}\{[\omega(1/t)]^{2m}+[\bar{\omega}(t)]^{2m}-2[\omega(1/t)\bar{\omega}(t)]^{m}\}\frac{dt}{t-\zeta}\Big]$$

$$-\frac{1+2\sigma}{8\pi i}\Big[\int_{\alpha}\{\omega(t)\bar{\omega}(1/t)\}^{m}\{[\omega(t)]^{m}-[\bar{\omega}(1/t)]^{m}\}\frac{dt}{t-\zeta}$$

$$-\int_{\beta}\{\omega(1/t)\bar{\omega}(t)\}^{m}\{[\omega(1/t)]^{m}-[\bar{\omega}(t)]^{m}\}\frac{dt}{t-\zeta}\Big]$$

$$+\frac{1+\sigma}{2\pi i}\left[\int_\alpha\{F(t)-\bar{F}(1/t)\}\frac{dt}{t-\zeta}-\int_\beta\{F(1/t)-\bar{F}(t)\}\frac{dt}{t-\zeta}\right]$$

$$-\frac{\sigma b(1-\cos 4\delta)}{4\pi i\sin 4\delta}\left[\int_\alpha\{[\omega(t)]^{2m}-[\bar{\omega}(1/t)]^{2m}\}\frac{dt}{t-\zeta}\right.$$

$$\left.-\int_\beta\{[\omega(1/t)]^{2m}-[\bar{\omega}(t)]^{2m}\}\frac{dt}{t-\zeta}\right]+\frac{(2\sigma-1)\sin 2\delta}{24\pi i\sin 6\delta}\left[\int_\alpha\{[\omega(t)]^{3m}-[\bar{\omega}(1/t)]^{3m}\}\frac{dt}{t-\zeta}\right.$$

$$\left.-\int_\beta\{[\omega(1/t)]^{3m}-[\bar{\omega}(t)]^{3m}\}\frac{dt}{t-\zeta}\right], \quad (3.10)$$

$$G_2(\zeta)=\frac{\sigma a}{4\pi}\left[\int_\alpha\{[\omega(t)]^{2m}+[\bar{\omega}(1/t)]^{2m}-2[\omega(t)\bar{\omega}(1/t)]^m\}\frac{dt}{t-\zeta}\right.$$

$$\left.-\int_\beta\{[\omega(1/t)]^{2m}+[\bar{\omega}(t)]^{2m}-2[\omega(1/t)\bar{\omega}(t)]^m\}\frac{dt}{t-\zeta}\right]$$

$$+\frac{\sigma a\tan 2\delta}{4\pi i}\left[\int_\alpha\{[\omega(t)]^{2m}-[\bar{\omega}(1/t)]^{2m}\}\frac{dt}{t-\zeta}-\int_\beta\{[\omega(1/t)]^{2m}-[\bar{\omega}(t)]^{2m}\}\frac{dt}{t-\zeta}\right]$$

$$-\frac{\sigma}{8\pi}\left[\int_\alpha\{[\omega(t)]^{3m}+[\bar{\omega}(1/t)]^{3m}-[\omega(t)\bar{\omega}(1/t)]^m([\omega(t)]^m+[\bar{\omega}(1/t)]^m)\}\frac{dt}{t-\zeta}\right.$$

$$\left.-\int_\beta\{[\omega(1/t)]^{3m}+[\bar{\omega}(t)]^{3m}-[\omega(1/t)\bar{\omega}(t)]^m([\omega(1/t)]^m+[\bar{\omega}(t)]^m)\}\frac{dt}{t-\zeta}\right]$$

$$+\frac{(1-2\sigma)\sin^2 2\delta\cos 2\delta}{6\pi i\sin 6\delta}\left[\int_\alpha\{[\omega(t)]^{3m}-[\bar{\omega}(1/t)]^{3m}\}\frac{dt}{t-\zeta}\right.$$

$$\left.-\int_\beta\{[\omega(1/t)]^{3m}-[\bar{\omega}(t)]^{3m}\}\frac{dt}{t-\zeta}\right]$$

$$-\frac{(1+\sigma)b}{2\pi i}\left[\int_\alpha\{G(t)-\bar{G}(1/t)\}\frac{dt}{t-\zeta}-\int_\beta\{G(1/t)-\bar{G}(t)\}\frac{dt}{t-\zeta}\right]$$

$$+\frac{1+\sigma}{8\pi}\left[\int_\alpha\{H(t)+\bar{H}(1/t)\}\frac{dt}{t-\zeta}-\int_\beta\{H(1/t)+\bar{H}(t)\}\frac{dt}{t-\zeta}\right]. \quad (3.11)$$

The expressions for $G_1(\zeta)$, $G_2(\zeta)$ are valid for all values of δ except $\pi/6$, $\pi/4$, $\pi/3$, $3\pi/4$.

The particular case of the sector of a circle is obtained by putting $\omega(\zeta)=\zeta$. The complex torsion and flexure functions for this case have been determined in a previous paper (Mitra, 1950).

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ON GENERATING FUNCTIONS OF POLYNOMIALS (I): GENERALISED PARABOLIC CYLINDER FUNCTIONS OF WEBER

By

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1. Let $D_{(k)}^{(k)}$ stand for the operator $\dfrac{d}{dy}\dfrac{1}{y^{k-2}}\dfrac{d}{dy}$ and let $D_{(k)}^{(kn)}$ denote the operator $D_{(k)}^{(k)}$ operating on a function n times successively.

Further let

$$\bar{D}_{km}(y) = \exp\{y^k/(2k)\}.D_{(k)}^{(km)}.\exp(-y^k/k)$$

and

$$\bar{D}_{km+1}(y) = -\exp\{y^k/(2k)\}.D_{(k)}^{(km+k-1)}.\exp(-y^k/k)$$

where $D_{(k)}^{(km+1)}$ stands for the operator $\dfrac{d}{dy}D_{(k)}^{(km)}$. We call $\bar{D}_{km}(y)$ and $\bar{D}_{km+1}(y)$ Generalised Parabolic Cylinder Functions.

In two earlier papers, (Mitra and Sharma, 1949) we have deduced the recurrence-relations and the orthogonal properties satisfied by these functions and also obtained many other properties. We have shown that

$$\bar{D}_{km}(y) = \frac{(-1)^m k^m \Gamma(m+1/k)}{\Gamma(1/k)}\, {}_1F_1(-m;\, 1/k;\, y^k/k).\exp\{-y^k/(2k)\},$$

and

$$\bar{D}_{km+1}(y) = \frac{(-1)^m k^m \Gamma(m+1+1/k)}{\Gamma(1+1/k)}\, y\, {}_1F_1(-m;\, 1+1/k;\, y^k/k).\exp\{-y^k/(2k)\}.$$

When $k = 2$, they degenerate into Weber's Parabolic Cylinder Functions. The polynomial parts degenerate into Laguerre Polynomials multiplied by constants, when we write x for y^k/k and $1/k = \alpha+1$ or $1/k = \alpha$ according as m is an even or an odd integer. If we denote the polynomial parts by $\bar{H}_{km}(y)$ and $\bar{H}_{km+1}(y)$, we find that

$$\bar{H}_{km}(y) = (-1)^m k^m \Gamma(m+1) L_m^{(1/k-1)}(y^k/k)$$

and

$$\bar{H}_{km+1}(y) = (-1)^m k^m \Gamma(m+1).y L_m^{(1/k)}(y^k/k)$$

Similarly Sonine's polynomials can be expressed in terms of our polynomials. It is sufficient to take $R(k) > 0$.

In this note we find generating functions for $\bar{H}_{km}(y)$ and $\bar{H}_{km+1}(y)$, analogous to the generating functions for Hermite and Laguerre Polynomials as deduced by Watson (1933). We also use a method of Gh. Th. Gheorghu to obtain a general generating function for these polynomials. We first give an independent proof based on Operational Calculus.

2. We can prove by direct term-by-term integration that

$$\exp(-x^k/k) = \frac{k}{\Gamma(1/k)}\int_0^\infty \exp(-u^k).{}_0F_1(1/k;\, -(x^k u^k)/k)\,du.$$

Operating on both sides by the operator $D_{(k)}^{(km)}$, we find that

$$D_{(k)}^{(km)} \exp(-x^k/k) = \frac{(-1)^m k^{m+1}}{\Gamma(1/k)} \int_0^\infty \exp(-u^k)\, u^{mk}.{}_0F_1(1/k;\; -(u^k x^k)/k)du.$$

Hence

$$\exp\{-x^k/(2k)\}.\bar{D}_{km}(x) = \frac{(-1)^m k^{m+1}}{\Gamma(1/k)} \int_0^\infty \exp(-u^k).u^{mk}.{}_0F_1(1/k;\; -(u^k x^k)/k)du.$$

Therefore

$$\sum_{m=0}^\infty \frac{\{\Gamma(1/k)\}^2 \exp\{-(x^k+y^k)/k\}.\bar{H}_{km}(x)\bar{H}_{km}(y).t^{mk}}{k^{2m+2}\Gamma(m+1)}, \quad (|t|<1)$$

$$= \sum_{m=0}^\infty \int_0^\infty \int_0^\infty \frac{\exp\{-(u^k+v^k)\}.(uvt)^{mk}}{\Gamma(m+1)} \times {}_0F_1(1/k;\; (u^k x^k)/k) \times {}_0F_1(1/k;\; -(v^k x^k)/k)dudv$$

$$= \int_0^\infty \int_0^\infty \frac{\exp(-u^k - v^k + t^k u^k v^k)}{\Gamma(m+1)} . {}_0F_1(1/k;\; -(u^k x^k)/k) \times {}_0F_1(1/k,\; -(v^k x^k)/k)dudv.$$

Now $$\int_0^\infty \exp\{-u^k(1-t^k v^k)\}.{}_0F_1(1/k;\; -(u^k x^k)/k)du$$

$$= \frac{\Gamma(1/k)}{k}.(1-t^k v^k)^{-1/k}.\exp\{-x^k/(k-kt^k v^k)\}.$$

Right hand side

$$= \frac{\Gamma(1/k)}{k} \int_0^\infty \exp\{-v^k - x^k/(k-kt^k v^k)\}.{}_0F_1(1/k;\; -(v^k y^k)/k) \times (1-t^k v^k)^{-1/k}.dv.$$

Therefore

$$\sum_{m=0}^\infty \frac{(-1)^m \Gamma(1/k)\exp\{-(x^k+y^k)/k\}}{k^{2m+1}.\Gamma(m+1)}.H_{km}(x)H_{km}(y).t^{mk+1-k}$$

$$= t^{1-k} \int_0^\infty \exp\{-v^k - x^k/(k-kt^k v^k)\}.{}_0F_1(1/k;\; -(v^k y^k)/k).(1-t^k v^k)^{-1/k}dv.$$

Let $t^k = 1/p$. Therefore

$$\sum_{m=0}^\infty \frac{\Gamma(1/k).\exp\{-(x^k+y^k)/k\}}{k^{2m+1}.\Gamma(m+1)}.H_{km}(x)H_{km}(y).\frac{1}{p^{m+1/k-1}}$$

$$= \int_0^\infty \exp\{-v^k - (px^k)/(kp-kv^k)\}.p(p-v^k)^{-1/k}.{}_0F_1(1/k;\; -(v^k y^k)/k)dv$$

Now if

$$f(p) \doteqdot h(s), \quad \text{then } p(p+a)^{-1}.f(p+a) \doteqdot \exp(-as)\, h(s)$$

Also

$$\exp(1/p).p^{-n} \doteqdot s^{\frac{1}{2}n}.I_n(2\sqrt{s}).$$

On interpretation, we find that the right-hand side is

$$= \Gamma(1/k).k^{1/k-1} \int_0^\infty \exp\{-x^k/k - v^k(1+s)\}.s^{\frac{1}{2}(1/k-1)}(xy)^{\frac{1}{2}(k-1)}.v^{k-1}$$
$$\times I_{1/k-1}(2(sx^k y^k/k)^{\frac{1}{2}}).J_{1/k-1}(2(y^k v^k/k)^{\frac{1}{2}}).dv.$$

Therefore writing $-s^2$ for s and using the known integral

$$\int_0^\infty \exp(-p^2z^2).J_\nu(az)J_\nu(bz)z\,dz = \{1/(2p^2)\}\exp\{-(a^2+b^2)/(4p^2)\}.I_\nu(ab/(2p^2)),$$

we get after a little simplification

$$\sum_{m=0}^\infty \frac{\exp\{-(x^k+y^k)/k\}.\bar{H}_{km}(x)\bar{H}_{km}(y).s^{2m+1/k-1}}{\Gamma(m+1/k)\Gamma(m+1).k^{2m}}$$
$$= (1-s^2)^{-1}.k^{1/k-1}(xy)^{\frac{1}{2}(k-1)}.\exp\{-(x^k+y^k)/(k-ks^2)\}\,I_{1/k-1}(2s(x^ky^k)^{\frac{1}{2}}.k^{-1}(1-s^2)^{-1}). \qquad (1)$$

3. Proceeding exactly as above we can prove that

$$\bar{D}_{km+1}(y) = \frac{(-1)^m k^{m+1}}{\Gamma(1/k+1)}\exp\{y^k/(2k)\}.y\int_0^\infty u^{(m+1)k}.\exp(-u^k)\,{}_0F_1(1/k+1;\,-(u^ky^k)/k)du.$$

Hence

$$\sum_{m=0}^\infty (-1)^m.\frac{\exp\{-(x^k+y^k)/k\}.\bar{H}_{km+1}(x)\bar{H}_{km+1}(y).t^{mk}}{\Gamma(m+1).k^{2m+2}}$$
$$= \frac{1}{\{\Gamma(1/k+1)\}^2}\int_0^\infty\int_0^\infty xy.u^kv^k.\exp(-u^k-v^k+t^ku^kv^k)$$
$$\times\,{}_0F_1(1/k+1;\,-(u^kx^k)/k)\,{}_0F_1(1/k+1;\,-(v^ky^k)/k)du\,dv.$$

Without going into the details of the calculation, we state that

$$\sum_{m=0}^\infty \frac{\exp\{-(x^k+y^k)/k\}.H_{km+1}(x)\bar{H}_{km+1}(y)s^{2n+1/k}}{k^{2n+1/k}.\Gamma(m+1)\Gamma(m+1/k)}$$
$$= (xy)^{\frac{1}{2}}.(1-s^2)^{-1}.\exp\{-(x^k+y^k)/(k-ks^2)\}.I_{1/k}(2s(x^ky^k)^{\frac{1}{2}}.k^{-1}(1-s^2)^{-1}) \qquad (2)$$

For $k = 2$, we get the formulae given by Watson, on adding (1) and (2).

4. We have proved that

$$\exp(-x^k/k)\bar{H}_{km}(x) = \frac{(-1)^m k^{m+1}}{\Gamma(1/k)}\int_0^\infty \exp(-u^k).u^{mk}.{}_0F_1(1/k;\,-(u^kx^k)/k)du.$$

Therefore

$$\Gamma(1/k)\sum_{m=0}^\infty \frac{(-1)^m\exp(-x^k/k).H_{km}(x).t^{mk}}{k^{m+1}\Gamma(m+1)}$$
$$= \int_0^\infty \exp(-u^k-t^ku^k).{}_0F_1(1/k;-(u^kx^k)/k)du = (1/k)\Gamma(1/k)(1+t^k)^{-1/k}.\exp\{-x^k/(k+kt^k)\}.$$

Let us put $t^k = 1/p$ and interpret. We get after a bit of simplification

$$\sum_{m=0}^\infty \frac{\bar{H}_{km}(x).s^{m+\frac{1}{2}(1/k-1)}}{k^{m+\frac{1}{2}(1/k-1)}\Gamma(m+1)\Gamma(m+1/k)} = x^{\frac{1}{2}(k-1)}\exp(-s)\,I_{1/k-1}(2(x^ks/k)^{\frac{1}{2}}). \qquad (3)$$

Similarly

$$\sum_{m=0}^\infty \frac{\bar{H}_{km+1}(x).s^{m+1/(2k)}}{k^{m+1/(2k)}\Gamma(m+1)} = x^{\frac{1}{2}}.\exp(-s).I_{1/k}(2(x^ks/k)^{\frac{1}{2}}). \qquad (4)$$

on putting $k=2$ and adding (3) and (4), we get Feldheim's result (1940), *viz.*

$$\sum_{r=0}^{\infty} (\omega^r/r!).H_r(x) = \exp\,(2\omega x-\omega^2) \tag{5}$$

5. We can easily shaw that

$$\sum_{m=0}^{\infty} \alpha^{km} C_{km} \bar{H}_{km}(x)$$

has its generating function given by

$$z(\alpha,\, x) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(1/k+1)\; (\alpha x)^{km}}{k^m \Gamma(m+1/k)\Gamma(m+1)} \,.\, F^{(m)}(\alpha^k) \tag{6}$$

where F is an arbitrary function and

$$C_{km} = \frac{(-1)^m \Gamma(1/k+1)}{k^m \Gamma(m+1/k)\Gamma(m+1)} F^{(m)}(0) \tag{7}$$

Similarly

$$\sum_{m=0}^{\infty} \alpha^{km+1}.C_{km+1}\bar{H}_{km+1}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(1/k+1)\,(\alpha x)^{km+1}}{k^m \Gamma(m+1/k+1)\Gamma(m+1)}.\Phi^{(m)}(\alpha^k) \tag{8}$$

where

$$C_{km+1} = \frac{(-1)^m \Gamma(1/k+1)}{k^m \Gamma(m+1/k+1)\Gamma(m+1)}.\Phi^{(m)}(0). \tag{9}$$

We shall now obtain a generating function $z(\alpha,\, x)$ for $\bar{H}_{km}(x)$, in the form $\xi(\alpha).\exp\{\zeta(\alpha)x^k\}$. It is easy to see that any such function $z(\alpha,\, x)$ satisfies the differential equation

$$\left(\frac{\partial}{\partial x}\frac{1}{x^{k-2}}\frac{\partial}{\partial x}\right)z - x\frac{\partial z}{\partial x} + \alpha\frac{\partial z}{\partial \alpha} = 0. \tag{10}$$

If $z(\alpha,\, x) = \xi(\alpha)\;\exp\{\zeta(\alpha)x^k\}$, we have on substituting in (10),

$$k\xi(\alpha).\zeta(\alpha) + \alpha\xi'(\alpha) = 0$$

$$k^2\zeta^2(\alpha) - k\zeta(\alpha) + \alpha\zeta'(\alpha) = 0$$

whence we have

$$\xi(\alpha) = b(\alpha^k+c)^{-1/k}. \quad \text{and} \quad \zeta(\alpha) = (\alpha^k/k)(\alpha^k+c)^{-1}.$$

On putting $x = 0$ and comparing with (6), we find

$$F(\alpha^k) = bk(\alpha^k+c)^{-1/k}$$

$$i.e \quad F(\alpha) = bk(\alpha+c)^{-1/k}.$$

Hence

$$C_{km} = \frac{(-1)^m \Gamma(1/k+1)bk}{k^m \Gamma(m+1/k)\Gamma(m+1)}.\left[\frac{d^m}{d\alpha^m}.(\alpha+c)^{-1/k}\right]_{\alpha=0} = \frac{c^{-m-1/k}.b}{k^m \Gamma(m+1)}.$$

so that

$$\sum_{m=0}^{\infty} \frac{c^{-m-1/k}.\alpha^{km}}{k^m \Gamma(m+1)} H_{km}(x) = (\alpha^k+c)^{-1/k}.\exp\{\alpha^k x^k/(k\alpha^k+kc)\}. \tag{11}$$

In order to find a similar term for the generating function of $\bar{H}_{km+1}(x)$, we assume

$$z(\alpha,\ x) = x\xi(\alpha).\exp\{\zeta(\alpha)x^k\} = \sum_{m=0}^{\infty} \alpha^{km+1} C_{km+1} \bar{H}_{km+1}(x).$$

Here $z(\alpha,\ x)$ satisfies the differential equation

$$\frac{\partial^2 z}{\partial a^2} - x^{\kappa-1}\frac{\partial z}{\partial x} + \alpha x^{k-2}\frac{\partial z}{\partial \alpha} = 0$$

whence we get

$$k(k+1)\xi(\alpha)\zeta(\alpha) - \xi(\alpha) + \alpha\xi\ (\alpha) = 0$$

$$k^2\zeta^2(\alpha) - k\zeta(\sigma) + \alpha\zeta'(\alpha) = 0$$

so that

$$\xi(\alpha) = b\alpha(\alpha^k + c)^{-1/k-1},\ \text{ and }\ \zeta(\alpha) = \alpha^k k^{-1}(\alpha^k + c)^{-1}.$$

Then

$$z(\alpha,\ x) = xb\alpha(\alpha^k + c)^{-1/k-1}\exp\{\alpha^k x^k k^{-1}(\alpha^k + c)^{-1}\}.$$

Comparing this with (8) and putting $x = 0$, after dividing by x, we see that

$$\Phi(\alpha) = b(\alpha + c)^{-1/k-1}.$$

Hence

$$C_{km+1} = \frac{c^{m-1/k-1}b}{k^m\Gamma(m+1)}.$$

so that

$$\sum_{m=0}^{\infty} \frac{c^{m-1/k-1}.\alpha^{km+1}.\bar{H}_{km+1}(x)}{k^m\Gamma(m+1)} = \alpha x(\alpha^k + c)^{-1/k-1}\exp\{\alpha^k x^k(\alpha^k + c)^{-1}\}. \tag{12}$$

We shall now only observe that we can also obtain relations (1), (2), (3) and (4) by adopting the method used to obtain (11) and (12).

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ON THERMODYNAMICS OF MATTER IN A STATIC FIELD

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1. Introduction.

The condition of thermodynamic equilibrium of a sphere of perfect fluid in a static gravitational field was obtained by Tolman (Tolman, 1934) in the form

$$T_0 \sqrt{g_{44}} = \text{const} \tag{1}$$

where T_0 is the proper temperature and g_{ik} is the metric tensor. In deriving this condition Tolman started from the condition of maximum entropy for thermodynamic equilibrium and made explicit use of Einstein's field equations. We shall, in future, refer to this as Tolman condition.

It is, however, possible to deduce the same condition starting from a different condition of thermodynamic equilibrium without use of any explicit form of gravitational field equations. The equation of heat conduction can be written in the form

$$-\kappa \,\mathrm{div}\,\mathrm{grad} T = c\varrho \frac{\partial T}{\partial t}$$

where c = specific heat, ϱ = density of the material medium, κ = its conductivity and T = temperature. In classical thermodynamics the condition of thermal equilibrium (T = const.) is obtained by putting the right hand side of the above equation equal to zero and integrating the resulting equation under the boundary condition of no flow of heat. Now this equation can be given a Lorentz-covariant form by the introduction of a "heat flow" vector thus taking it over into relativity. Then proceeding as in classical thermodynamics we can deduce the condition (1) without explicit use of gravitational field equations.

2. Derivation of Tolman Condition.

Let us introduce the "heat flow" four-vector by its covariant components

$$\Gamma_\mu = \kappa \frac{\partial T}{\partial x^1},\ \kappa \frac{\partial T}{\partial x^2},\ \kappa \frac{\partial T}{\partial x^3},\ c\varrho T \tag{2}$$

where T is the co-ordinate temperature. Since the special relativistic line-element is

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2$$

the contravariant components of the vector is

$$\Gamma^\mu = -\kappa \frac{\partial T}{\partial x^1},\ -\kappa \frac{\partial T}{\partial x^2},\ -\kappa \frac{\partial T}{\partial x^3},\ c\varrho T$$

Hence the equation of heat conduction can be written in the form

$$\frac{\partial \Gamma^\mu}{\partial x^\mu} = 0$$

This form is obviously Lorentz covariant and hence can be taken as the special relativistic equation of heat conduction. To get the general relativistic form we appeal to the two fundamental principles of general relativity, *viz.*, (1) principle of general covariance, (2) principle of equivalence. We choose

$$\Gamma^\mu{}_{;\mu} \tag{3}$$

as the required equation of heat conduction, where (;) denotes covariant differentiation. Because, firstly, this equation is covariant with respect to general co-ordinate transformations, and secondly, in the neighbourhood of a point where it is possible to introduce Galelian frame of reference the equation reduces to the special relativistic form.

Equation (3) can be written as

$$\frac{\partial}{\partial x^\mu} (\Gamma^\mu \sqrt{(-g)}) = 0 \tag{3$'$}$$

Now in the general Riemann space let us consider a line-element of the form

$$ds^2 = -g_{11}(dx^1)^2 - g_{22}(dx^2)^2 - g_{33}(dx^3)^2 + g_{44}(dx^4)^2$$

In obvious cases of symmetry the line-elements will certainly take this form. Since the gravitational field is assumed to be static, the metric tensor will be independent of the time co ordinate x^4.

In this field let us consider a medium in a steady state of temperature. The equation (3′) reduces to

$$\frac{\partial}{\partial x^1}\left(g^{11}\sqrt{(-g)}\frac{\partial T}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(g^{22}\sqrt{(-g)}\frac{\partial T}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(g^{33}\sqrt{(-g)}\frac{\partial T}{\partial x^3}\right) = 0 \tag{3a}$$

From this we have

$$\iiint \left[\frac{\partial}{\partial x^1}\left(g^{11}\sqrt{(-g)}\,\frac{\partial T}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(g^{22}\sqrt{(-g)}\frac{\partial T}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(g^{33}\sqrt{(-g)}\,\frac{\partial T}{\partial x^3}\right)\right] dx^1 dx^2 dx^3 = 0$$

$$\text{or, } \iint \left[\left(g^{11}\sqrt{(-g)}.\,\frac{\partial T}{\partial x^1}\right) dx^2 dx^3 + \left(g^{22}\sqrt{(-g)}\,\frac{\partial T}{\partial x^2}\right) dx^3 dx^1 + \left(g^{33}\sqrt{(-g)}.\frac{\partial T}{\partial x^3}\right) dx^1 dx^2\right] = 0$$

$$\text{or. } \iint \sqrt{g_{44}}\left[\frac{\partial T}{\partial x^1}\sqrt{(g_{22}g_{33}g^{11})}dx^2dx^3 + \frac{\partial T}{\partial x^2}\sqrt{(g_{33}g_{11}g^{22})}dx^3dx^1 + \frac{\partial T}{\partial x^3}\sqrt{(g_{11}g_{22}g^{33})}dx^1dx^2\right] = 0$$

Let us now consider an element dS of the surface S enclosing the domain of integration. Let ν be the unit vector normal to dS. Then as shown by Levi-civita (Levi-civita, 1929)

$$\nu^1 = (\vec{\nu}.\vec{x}^1)\sqrt{g^{11}} = \cos\widehat{(\nu x^1)}\sqrt{g^{11}}$$

$$\therefore\quad dS\nu^1 = dS\cos\widehat{(\nu x^1)}\sqrt{g^{11}} = \sqrt{(g^{11}g_{22}g_{33})}dx^2dx^3.$$

Similarly,

$$dS\nu^2 = \sqrt{(g_{33}g_{11}g^{22})}dx^3dx^1,\ dS\nu^3 = \sqrt{(g_{11}g_{22}g^{33})}dx^1dx^2.$$

Hence the above equation reduces to

$$\iint \sqrt{g_{44}}\left(\frac{\partial T}{\partial x^1}\nu^1+\frac{\partial T}{\partial x^2}\nu^2+\frac{\partial T}{\partial x^3}\nu^3\right)dS=0$$

$$i.e.,\ \iint \sqrt{g_{44}}\,\frac{\partial T}{\partial \nu}\,dS=0 \tag{3b}$$

So this is the condition to be satisfied at the boundary in order that (3a) may be satisfied throughout the region This condition may be interpreted as: the total flux of heat energy into the region across the surface is zero. For, if we refer to proper co-ordinates, (3b) reduces to

$$\iint \frac{\partial T_0}{\partial \nu}\,dS=0$$

(where T_0 is the proper temperature) which is, indeed, the condition that the net flux of heat energy into the region is zero. The interpretation is possible even without reference to proper co-ordinates. For, $\partial T/\partial \nu$ is the flux of heat energy at a point of the surface per unit time. Hence the flux across a surface element dS for the co-ordinate time interval dt *i.e.*, the observer's time intervals $\sqrt{g_{44}}dt$ is

$$\frac{\partial T}{\partial \nu}\,dS.\sqrt{g_{44}}dt$$

Hence the total flux across the whole surface in the said interval is

$$dt\iint \frac{\partial T}{\partial \nu}\sqrt{g_{44}}dS$$

and this is equal to zero by (3a). This is the case of the medium being in a steady state. In the state of static thermodynamic equilibrium, however, there is no exchange of energy between the interior and the exterior at any point of the surface. Hence in this case

$$\frac{\partial T}{\partial \nu}=0 \tag{3c}$$

everywhere on the surface.

Next let us consider the integral

$$\iiint T\left[\frac{\partial}{\partial x^1}\left(g^{11}\sqrt{(-g)}\,\frac{\partial T}{\partial x^1}\right)+\frac{\partial}{\partial x^2}\left(g^{22}\sqrt{(-g)}\,\frac{\partial T}{\partial x^2}\right)+\frac{\partial}{\partial x^3}\left(g^{33}\sqrt{(-g)}\,\frac{\partial T}{\partial x^3}\right)\right]dx^1dx^2dx^3$$

$$=\iint T\sqrt{g_{44}}\left[\frac{\partial T}{\partial x^1}\sqrt{(g_{22}g_{33}g^{11})}dx^2dx^3+\frac{\partial T}{\partial x^2}\sqrt{(g_{33}g_{11}g^{22})}dx^3dx^1+\frac{\partial T}{\partial x^3}\sqrt{(g_{11}g_{22}g^{33})}dx^1dx^2\right]$$

$$-\iiint \sqrt{(-g)}\left[g^{11}\left(\frac{\partial T}{\partial x^1}\right)^2+g^{22}\left(\frac{\partial T}{\partial x^2}\right)^2+g^{33}\left(\frac{\partial T}{\partial x^3}\right)^2\right]dx^1dx^2dx^3$$

$$=\iint \sqrt{g_{44}}\,\frac{\partial T}{\partial \nu}\,dS-\iiint \sqrt{g_{44}}\left[g^{11}\left(\frac{\partial T}{\partial x^1}\right)^2+g^{22}\left(\frac{\partial T}{\partial x^2}\right)^2+g^{33}\left(\frac{\partial T}{\partial x^3}\right)^2\right]d\tau$$

where $d\tau$ is the three-dimensional volume element.

Now the integral on the left is zero by (3a). The surface integral on the right is zero by (3c). Hence

$$\iiint \left[g^{11}\left(\frac{\partial T}{\partial x^1} \right)^2 + g^{22}\left(\frac{\partial T}{\partial x^2} \right)^2 + g^{33}\left(\frac{\partial T}{\partial x^3} \right)^2 \right] \sqrt{g_{44}}\, d\tau = 0$$

The coefficients g^{11}, g^{22}, g^{33} are all positive ; Hence the integrand is non-negative at every point of the volume. So since the integral vanishes, the integrand which is continuous everywhere, must vanish at every point of the volume *i.e.*,

$$g^{11}\left(\frac{\partial T}{\partial x^1} \right)^2 + g^{22}\left(\frac{\partial T}{\partial x^2} \right)^2 + g^{33}\left(\frac{\partial T}{\partial x^3} \right)^2 = 0$$

Again since g^{11}, g^{22}, g^{33} are positive this means

$$\frac{\partial T}{\partial x^1} = \frac{\partial T}{\partial x^2} = \frac{\partial T}{\partial x^2} = 0$$

throughout the volume. Hence

$$T = \text{const. (throughout the volume)} \tag{4}$$

This T, however, is the co-ordinate temperature. To find its relation with proper temperature we start with the corresponding relation in special relativity

$$T = T_0 \sqrt{(1-\beta^2)}$$

where
$$\beta^2 = \left(\frac{dx^1}{dx^4}\right)^2 + \left(\frac{dx^2}{dx^4}\right)^2 + \left(\frac{dx^3}{dx^4}\right)^2$$

Here
$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2$$

$$\therefore \quad \left(\frac{ds}{dx^4}\right)^2 = 1 - \beta^2$$

$$\therefore \quad T_0 = T\frac{dx^4}{ds}$$

We assume that the same relation holds in general relativity. We have assumed the line-element to be of the form

$$ds^2 = -g_{11}(dx^1)^2 - g_{22}(dx^2)^2 - g_{33}(dx^3)^2 + g_{44}(dx^4)^2$$

$$\therefore \quad \left(\frac{ds}{dx^4}\right)^2 = g_{44}$$

since $\dfrac{dx^i}{dx^4} = 0$ $(i = 1, 2, 3)$, the medium being at rest.

$$\therefore \quad T = T_0\frac{ds}{dx^4} = T_0\sqrt{g_{44}}$$

Hence (4) reduces to

$$T_0\sqrt{g_{44}} = \text{const.}$$

which is Tolman's condition.

Next, instead of static thermodynamic equilibrium, let the medium be in a steady state. Let the boundary of the medium be maintained at the same constant co-ordinate temperature T throughout. Then

$$\iiint T\left[\frac{\partial}{\partial x^1}\left(g^{11}\sqrt{(-g)}.\frac{\partial T}{\partial x^1}\right)+\frac{\partial}{\partial x^2}\left(g^{22}\sqrt{(-g)}.\frac{\partial T}{\partial x^2}\right)+\frac{\partial}{\partial x^3}\left(g^{33}\sqrt{(-g)}.\frac{\partial T}{\partial x^3}\right)\right]dx^1dx^2dx^3$$

$$=T\iint \sqrt{g_{44}}\frac{\partial T}{\partial \nu}.\,dS-\iiint\left[g^{11}\left(\frac{\partial T}{\partial x^1}\right)^2+g^{22}\left(\frac{\partial T}{\partial x^2}\right)^2+g^{33}\left(\frac{\partial T}{\partial x^3}\right)^2\right]\sqrt{g_{44}}d\tau$$

As before, the integral on the left vanishes by (3a). The surface integral on the right vanishes by (3b). Hence as before

$$\iiint\left[g^{11}\left(\frac{\partial T}{\partial x^1}\right)^2+g^{22}\left(\frac{\partial T}{\partial x^2}\right)^2+g^{33}\left(\frac{\partial T}{\partial x^3}\right)^2\right]\sqrt{g_{44}}d\tau=0$$

The same discussion as before, now shows that

$$T=\text{const.}$$

$$i.e.,\qquad T_0\sqrt{g_{44}}=\text{const.}$$

It has been assumed that the boundary condition has been so adjusted that $T_0\sqrt{g_{44}}=$ constant on the boundary. From the continuity of $T_0\sqrt{g_{44}}$ it follows that the value of the constant is the same on the boundary as inside and the boundary condition can be attained

3. Conclusion.

In the above discussion we have started from a different assumption for thermodynamic equilibrium from Tolman's and have deduced the condition without explicit use of any form of field equations so that the condition remains valid even if alternative field equations are proposed In fact, the condition will remain valid as long as the geometry of space is assumed Riemannian. Also, by avoiding the field equation, we have avoided any sssumption about the structure of the material medium since the stress-energy tensor T_{ik} appears nowhere Moreover, it has been shown that Tolman's condition holds in a steady state if the boundary condition is properly adjusted.

In conclusion, I wish to thank Prof. N. R. Sen for his helpful guidance at every stage of the work.

Department of Applied Mathematics
University of Calcutta.

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Levi-Civita, T. (1929), *The Absolute Differential Calculus* (English trans.). p. 128.
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CALCUTTA MATHEMATICAL SOCIETY

Report of the Council for the year 1950 to the Annual General Meeting of the Society

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1950 as required by the provisions of Rule 25.

The Council: The Council of the Society for the year 1950 consisting of the officers and other members elected at the last Annual General Meeting and co-opted thereafter, together with the Editorial Secretary, was constituted as follows:

President

Professor M. R. Siddiqi

Vice-Presidents

Prof. F. W. Levi — Prof. B. B. Sen

Prof. V. V. Narlikar — Prof. A. C. Banerjee

Dr. S. K. Chakraborty

Treasurer

Mr. S. C. Ghosh

Secretary

Mr. U. R. Burman

Editorial Secretary

Mr. P. K. Ghosh

Other Members of the Council

Dr. S. S. Pillai	Prof. C. N. Srinivasiengar
Dr. R. N. Sen	Mr. B. N. Mukherjee
Dr. T. Vijayaraghavan	Dr. N. G. Shabde
Mr. S. Gupta	Prof. N. M. Basu
Prof. N. R. Sen	Dr. A. C. Choudhury
Dr. S. Ghosh	Dr. P. L. Bhatnagar

Dr. S. S. Bhatnagar (Co-opted, representative of the Government of India).

General: The various activities of the Society have been carried on throughout the year in much the same form as in the past few years. The Council has the pleasure to report that the United States Vice-Consul in Calcutta was kind enough to pay a visit to the Society with a view to present personally some scientific papers from the United States Atomic Energy Commission. The Council offers its sincere thanks to the Vice-Consul for his personal visit and to the American Atomic Energy Commission for these valuable publications.

Membership: The Council records with regret the death of one Honorary and two ordinary members of the Society during the year under review. They are

Prof. C. Carathéodory (Hon. Member)
Dr. S. S. Pillai
Mr. Prasanta Kumar Chatterjee.

The death of Dr. Pillai occuring under most tragic circumstances in an air crash while on his way to the United States, has been a severe loss to the Society. The Council recalls with grateful appreciation his services as a former Vice-President and a member of the Editorial Board of the Society's Bulletin, and conveys its profound sense of sorrow to the members of the bereaved family.

The Council also reports that there has been a number of additions to the list of ordinary members of the Society, seven new members being elected during the year.

Meetings During 1950: The Council held six meetings during the year and there were five ordinary general meetings devoted to the reading of original papers communicated to the Society for publication in its Bulletin.

Publications: During the year under review the Society has published three numbers of the Bulletin namely, Vol. 42, Nos. 1-3. The Council again records the the Society's deep indebtedness to the authorities of the Calcutta University for printing the Bulletin free of charge and to the officers and members of the staff of the University Press who have made every endeavour to bring out the Bulletin in time, inspite of conditions in the printing industry being still very difficult.

Exchange of Publications: The transmission of the Society's publications to various countries in the world has been carried on regularly during the year and some new exchange relations have also been established. It is a matter of satisfaction that the society is now in exchange relation with almost all the mathematical and physico-mathematical societies of the world.

Library: It was mentioned in the previous year's Council report that the usefulness of the Society's Library had been considerably enhanced by the addition to the existing list of some periodicals which could not be obtained on an exchange basis and it was also stated that a capital grant from the Government of India made this possible. The Council is glad to report that out of this grant three more journals, namely Astronomische Nachrichten, Helvetica Physica Acta and Transactions of the American Mathematical Society have also been purchased during the year under review.

The construction of the steel shelves undertaken in the latter part of the previous year has been completed at a cost of about Rs. 1600/- and it is hoped that the additional shelf-space thus provided for the library will relieve the congestion for some years to come.

Finance: The annual accounts of the Society for the year 1950 have been presented to the Council in the standardised form by the auditors Dr. S. K. Basu and Mr. N. L. Ghosh. The Council gratefully acknowledges its indebtedness to them for their honorary services.

A comparison of the receipts and disbursements accounts would show an apparent deficit for the year. It may however be pointed out that the amount under the head

books and journals was available from last year's capital grant of the Government of India. This would account for this deficit.

The Society received a grant of Rs. 1000/- from the Government of West Bengal and a grant of Rs. 500/- from the National Institute of Sciences of India during the year under review, and the Council has the pleasure to report that these grants have enabled the Society to maintain the improvements in the Bulletin. The Council takes this opportunity to offer its grateful thanks to the Government of West Bengal and the National Institute of Sciences of India for these grants.

Office Staff: The Council regrets to report that the Society's bearer Ramgolam Ram who had been on long leave on account of illness died in November last The deceased rendered a long period of honest and faithful service which the Council sincerely appreciates.

CALCUTTA MATHEMATICAL SOCIETY

Receipts and Disbursements Accounts of the Calcutta Mathematical Society for the year ending 31st December, 1950.

Receipts

			Rs. as. p.
1. Opening balance			
(a) With Secretary			
(i) In cash ...	3 6 6		
(ii) In stamps ...	3 13 0	7 3 6	
(b) Balance at Banks			
(i) Imperial Bank of India	5,149 6 9		
(ii) Do. (K. K. G. P. Fund)	457 1 8		
(iii) United Bank of India	751 13 2		
,, In suspense	12 0 0		
(iv) P. O. Savings Bank	750 8 0	7,120 13 7	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000) ...		5,663 11 6	
(d) G. P. Notes (K.K.G.P. Fund) (Face value Rs. 2,000) ...		1,937 7 9	14,729 4 4
2. Membership subscription ...		838 14 0	
3. Admission fees ...		60 0 0	
4. Sale Proceeds ...		2,074 9 0	
5 Grants			
(i) West Bengal Government		1,000 0 0	
(ii) Nat. Inst. Sciences .		500 0 0	4,473 7 0
6 Interest			
(a) G. P. Notes (General Fund) . .		180 0 0	
(b) Do. (K. K. G. P. Fund) ...		60 0 0	
(c). P. O. Savings Bank ...		6 0 0	246 0 0
TOTAL Rs			19,448 11 4

Disbursements

				Rs. as. p.
1. Establishment				
(a) General	1,088 0 0			
(b) Construction of steel racks	1,570 0 0	2,658 0 0		
2 Meetings ...		91 15 0		
3. Books & Journals (including binding charges)		2,798 6 6		
4. Bulletins				
(a) Papers, Blocks and Types	634 15 9			
(b) Postage	270 13 0	905 12 9		
5. Printing and Stationery ...		90 1 9		
6. Postage (General)		41 6 6		
7. Bank charges ...		12 10 0		
8. Miscellaneous (including conveyance charges)		48 14 0		6,647 2 6
Closing Balance				
(a) With Secretary				
(i) In Cash	12 5 0			
(ii) In stamps	3 0 3	15 5 3		
(b) Balance at Banks				
(i) Imperial Bank of India	2,377 0 6			
(ii) Do (K. K. G. P. Fund)	516 9 8			
(iii) United Bank of India	1,528 6 2			
,, In suspense	6 8 0			
(iv) P O Savings Bank	756 8 0	5,185 0 4		
(c) G. P. Notes (General Fund) (Face value Rs. 6,000) ...		5,663 11 6		
(d) G. P. Notes (K. K. G. P. Fund) (Face value Rs. 2,000) ...		1,937 7 9	12,801 8 10	
TOTAL ... Rs.				19,448 11 4

To

The MEMBERS of the Calcutta Mathematical Society,

We have examined the above Balance Sheet with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

S K BASU
N. L GHOSH *Auditors*

All correspondence with the Society, subscriptions to the Bulletin, admission fees and annual contributions of members are to be sent to the *Secretary, Calcutta Mathematical Society, 92, Upper Circular Road, Calcutta 9.*

Papers intended for publication in the Bulletin of the Society, and all Editorial Correspondence, should be addressed to the *Editorial Secretary, Calcutta Mathematical Society.*

The publications of the Calcutta Mathematical Society may be purchased direct from the Society's office, or from its agents—Messrs. Bowes & Bowes, Booksellers and Publishers, 1, Trinity Street, Cambridge, England.

NOTICE TO AUTHORS

The manuscript of each paper communicated for publication in this Journal should be legibly written (preferably type-written) on one side of the paper and should be accompanied by a short *abstract* of the paper.

References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of the paper, the author's name following in alphabetical order, and each reference is to be arranged in the following form, *viz.*, name or names of authors ; year of publication ; name of the journal (abbreviation) ; number of volume ; and lastly, the page number. The following would be a useful illustration :

Wilson, B. M. (1922), *Proc. Lond. Math. Soc.* (2), 21, 235.

Authors of papers printed in the Bulletin are entitled to receive, free of cost, 50 separate copies of their communications. They can, however, by previous notice to the Secretary, ascertain whether it will be possible to obtain more copies even on payment of the usual charges.

CONTENTS

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ON CERTAIN RELATIONS BETWEEN ULTRASPHERICAL POLYNOMIALS AND BESSEL FUNCTIONS

By

A. Sharma, *Lucknow*

(Communicated by Dr S. C. Mitra—Received November 6, 1950)

1. Recently S. C. Mitra (1935, 1936, 1939) has obtained certain relations between Legendre Polynomials and Bessel Functions. Other interesting relations were later obtained by B. N. Bose and S. K. Bose. The object of this note is to generalise some of these results by replacing the Legendre Polynomials by Ultraspherical Polynomials. We use these results to obtain certain generalisations of integrals involving Bessel Functions obtained by B. N. Bose (1944). We do not obtain here the analogous generalisations of the results of S. K. Bose (1946), but they can be easily obtained.

2. Here we use the following Rodrigue's analogue for $P_n^\lambda(x)$.

$$P_n^\lambda(x) = \frac{(-2)^n\Gamma(n+\lambda)\Gamma(n+2\lambda)}{n\,!\Gamma(2n+2\lambda)\Gamma(\lambda)}(1-x^2)^{\frac{1}{2}-\lambda}\frac{d^n}{dx^n}\{(1-x^2)^{n+\lambda-\frac{1}{2}}\}.$$

This reduces to Legendre Polynomials for $\lambda = \frac{1}{2}$. For $\lambda = 0$ and 1, we get

$$P_n^0(x) = \cos n\theta, \quad P_n^1(x) = \sin\,(n+1)\theta/\sin\theta,$$

where $x = \cos\theta$. We shall throughout consider $\lambda > -\frac{1}{2}$ (unless otherwise stated) and n a positive integer.

We shall have occasion to use the following integral very often.

$$I = \int_0^1 P_n^\lambda(1-2y^2)(1-y^2)^{\lambda-\frac{1}{2}}y^{2\lambda+2r+2\nu}dy, \quad (\lambda+r+\nu > -\tfrac{1}{2}).$$

In order to evaluate this, we put $1-2y^2 = x$ and we find after a little simplification,

$$I = \frac{\pi^{\frac{1}{2}}(-1)^n\Gamma(n+2\lambda)\Gamma(\nu+r+1)\Gamma(\nu+r+\lambda+\frac{1}{2})}{2^{2\lambda}n\,!\Gamma(\lambda)\Gamma(\nu+r-n+1)\Gamma(\nu+r+n+2\lambda+1)}. \qquad (2.1)$$

On using (2.1), we can easily deduce the following results :

$$\int_0^1 P_n^\lambda(1-2y^2).J_\lambda(2yz).y^\lambda.(1-y^2)^{\lambda-\frac{1}{2}}dy = \frac{\pi\Gamma(n+2\lambda)}{2^{2\lambda}.n\,!\Gamma(\lambda)}.z^{-\lambda}.J^2_{n+\lambda}(z) \qquad (2.2)$$

$$\int_0^1 P_n^\lambda(1-2y^2).J_{-\lambda}(2yz).y^\lambda.(1-y^2)^{\lambda-\frac{1}{2}}dy = \frac{(-1)^n.\pi\Gamma(n+2\lambda)}{2^{2\lambda}.n\,!\Gamma(\lambda)}.z^{-\lambda}.J_{n+\lambda}(z)J_{-n-\lambda}(z) \qquad (2.3)$$

$$\int_0^1 P_n^\lambda(1-2y^2).J_{\lambda-\frac{1}{2}}(yz).y^{\lambda+\frac{1}{2}}.(1-y^2)^{\lambda-\frac{1}{2}}dy = \frac{\pi^{\frac{1}{2}}\Gamma(n+2\lambda)}{n\,!\Gamma(\lambda).2^{\lambda-\frac{1}{2}}}.z^{-\lambda-\frac{1}{2}}.J_{2n+2\lambda}(z) \qquad (2.4)$$

$$\int_0^1 P_n^\lambda(1-2y^2).H_{\lambda-\frac{1}{2}}(yz).y^{\lambda+\frac{1}{2}}.(1-y^2)^{\lambda-\frac{1}{2}}dy$$

$$= \frac{\pi^{\frac{1}{2}}\Gamma(n+2\lambda)}{\Gamma(\lambda).2^{2\lambda}\,n\,!}\left[\left(\frac{z}{2}\right)^{-\lambda-\frac{1}{2}}.H_{2n+2\lambda}(z)+\sum_{m=0}^{n-1}\frac{(-1)^{m+n}\,(\frac{1}{2}z)^{2m+\lambda+\frac{1}{2}}}{\Gamma(m-n+\frac{3}{2})\Gamma(m+n+2\lambda+\frac{3}{2})}\right] \qquad (2.5)$$

And $$\int_0^1 P_n^{\lambda}(1-2y^2).K_{\lambda}(2yz).y^{\lambda}.(1-y^2)^{\lambda-\frac{1}{2}}dy = \frac{\pi\Gamma(n+2\lambda)}{n\,!2^{2\lambda}.\Gamma(\lambda)}z^{-\lambda}.I_{n+\lambda}(z).K_{n+\lambda}(z) \tag{2.6}$$

Multiplying both sides of (2.4) by $z^{-\lambda+\frac{1}{2}}$ and differentiating with respect to z, we have

$$\int_0^1 P_n^{\lambda}(1-2y^2).y^{\lambda+\frac{3}{2}}.J_{\lambda+\frac{1}{2}}(yz).(1-y^2)^{\lambda-\frac{1}{2}}dy$$

$$= \frac{\Gamma(n+2\lambda).\pi^{\frac{1}{2}}}{2^{\lambda-\frac{1}{2}}.n\,!\Gamma(\lambda)}\cdot\frac{z^{-\lambda-\frac{1}{2}}}{2(n+\lambda)}\left[(n+2\lambda)J_{2n+2\lambda+1}(z)-nJ_{2n+2\lambda-1}(z)\right]. \tag{2.7}$$

Multiply both sides of (2.7) by $z^{\lambda+\frac{1}{2}}$ and differentiate with respect to z; then we have

$$\int_0^1 P_n^{\lambda}(1-2y^2)J_{\lambda-\frac{1}{2}}(yz).y^{\lambda+\frac{5}{2}}.(1-y^2)^{\lambda-\frac{1}{2}}dy$$

$$= \frac{\Gamma(n+2\lambda).\pi^{\frac{1}{2}}}{2^{\lambda-\frac{1}{2}}.n\,!\Gamma(\lambda)}\cdot\frac{z^{-\lambda-\frac{1}{2}}}{2(n+\lambda)}\left[(n+2\lambda)J'_{2n+2\lambda+1}(z)-nJ'_{2n+2\lambda-1}(z)\right] \tag{2.8}$$

More generally we find that

$$\int_0^1 P_n^{\lambda}(1-2y^2).J_m(yz).y^{m+2\lambda}.(1-y^2)^{\lambda-\frac{1}{2}}dy, \quad (m+\lambda > -\tfrac{1}{2})$$

$$= \frac{(-1)^n\Gamma(n+2\lambda)\pi^{\frac{1}{2}}}{2^{2\lambda}.n\,!\Gamma(\lambda)}\cdot\frac{\Gamma(\lambda+\frac{1}{2}+m)}{\Gamma(m-n+1)\Gamma(m+n+1+2\lambda)}\cdot(\tfrac{1}{2}z)^m$$
$$\times {}_1F_2(m+\lambda+\tfrac{1}{2};\ m-n+1,\ m+n+1+2\lambda;\ -\tfrac{1}{4}z^2).$$

$$= \frac{(-1)^n\Gamma(n+2\lambda)\pi^{\frac{1}{2}}\Gamma(m+\lambda+\frac{1}{2})}{2^{2\lambda}.n\,!\Gamma(\lambda)}(2/z)^{\lambda+\frac{1}{2}}$$

$$\times\sum_{r=0}^{\infty}\frac{(-n-\lambda+\frac{1}{2})_r(n+\lambda+\frac{1}{2})_r\Gamma(m+\frac{1}{2}+\lambda+r)(m+\frac{1}{2}+\lambda+2r)}{r\,!\Gamma(m-n+1+r)\Gamma(m+n+1+2\lambda+r)}\times J_{m+\frac{1}{2}+\lambda+2r}(z) \tag{2.9}$$

The formula (2.9) is obtained on using formula of B. N. Bose (1944).

If $m-n$ is a negative integer, we easily have from (2.9),

$$\int_0^1 P_n^{\lambda}(1-2y^2)J_m(yz)y^{m+2\lambda}.(1-y^2)^{\lambda-\frac{1}{2}}dy, \quad (m+\lambda > -\tfrac{1}{2})$$

$$= \frac{(-1)^m\Gamma(n+2\lambda)\pi^{\frac{1}{2}}}{2^{2\lambda}.n\,!\Gamma(\lambda)}(2/z)^{\lambda+\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(m+\lambda+\frac{1}{2})\Gamma(r+2n-m+\lambda+\frac{1}{2})\Gamma(r+n+\lambda+\frac{1}{2})}{\Gamma(m+\frac{1}{2}+\lambda-r)\Gamma(r+1)\Gamma(r+n-m+1)\Gamma(r+2n+2\lambda+1)}$$
$$\times(2r+2n-m+\lambda+\tfrac{1}{2})J_{2r+2n-m+\lambda+\frac{1}{2}}(z). \tag{2.10}$$

In a similar manner we have for all real values of m,

$$\int_0^1 P_n^{\lambda}(1-2y^2)J_m(yz).y^{2\lambda-m}.(1-y^2)^{\lambda-\frac{1}{2}}dy = \frac{(-1)^n\Gamma(n+2\lambda)\pi^{\frac{1}{2}}}{2^{2\lambda}.n\,!\Gamma(\lambda)}(2/z)^{\lambda+\frac{1}{2}}$$

$$\times\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}-\lambda+m+r)\Gamma(n+\frac{1}{2}+\lambda+r)\Gamma(m+2n+\lambda+\frac{1}{2}+r)}{\Gamma(\frac{1}{2}-\lambda+m)r\,!\Gamma(m+n+1+r)\Gamma(2n+1+2\lambda+r)}(m+2n+\lambda+\tfrac{1}{2}+2r)$$
$$\times J_{m+2n+\lambda+\frac{1}{2}+2r}(z). \tag{2.11}$$

3. *Relation between Whittaker Function and Ultraspherical Polynomials.*

By using (2.1) we can show that if $2m+1$ is not a negative integer,

$$\int_0^1 P_n^\lambda(1-2y^2).(1-y^2)^{\lambda-\frac{1}{2}}.(yz)^{-2m+2\lambda-1}\exp(-\tfrac{1}{2}y^2z^2).M_{k,m}(y^2z^2)\,dy$$

$$= \frac{(-1)^n\pi^{\frac{1}{2}}\Gamma(n+2\lambda).\Gamma(2m+1)\Gamma(m+\frac{1}{2}+k+n)\Gamma(\lambda+\frac{1}{2}+n)}{2^{2\lambda}.n\,!\Gamma(\lambda).\Gamma(m+\frac{1}{2}+k)\Gamma(2n+1+2\lambda)\Gamma(2m+1+n)}z^{2n+2\lambda}$$

$$\times {}_2F_2(m+\tfrac{1}{2}+k+n,\ \lambda+\tfrac{1}{2}+n;\ 2m+1+n,\ 2n+1+2\lambda;\ -z^2) \qquad (3.1)$$

since in this case we can make use of Kummer's first formula. Putting $m = \frac{1}{2}\lambda-\frac{1}{4}$, we have

$$\int_0^1 P_n^\lambda(1-2y^2).(1-y^2)^{\lambda-\frac{1}{2}}(yz)^{\lambda-\frac{1}{2}}.M_{k,\frac{1}{2}\lambda-\frac{1}{4}}(y^2z^2)\,dy$$

$$= \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)(-1)^n\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}\lambda+\frac{1}{4}+k+n)}{\Gamma(\lambda).2^{2\lambda}.n\,!\Gamma(\frac{1}{2}\lambda+\frac{1}{4}+k)\Gamma(2n+1+2\lambda)}\exp(-\tfrac{1}{2}z^2).z^{-1}\times M_{k-\frac{1}{2}\lambda-\frac{1}{4},n+\lambda}(z^2). \qquad (3.2)$$

Similarly, putting $k = n+2\lambda+\frac{1}{2}-m$, we have

$$\int_0^1 P_n^\lambda(1-2y^2).(1-y^2)^{\lambda-\frac{1}{2}}.(yz)^{-2m+2\lambda-1}.\exp(-\tfrac{1}{2}y^2z^2).M_{n+2\lambda+\frac{1}{2}-m,m}(y^2z^2)\,dy$$

$$= \frac{(-1)^n\pi^{\frac{1}{2}}\Gamma(n+2\lambda)\Gamma(2m+1)\Gamma(n+\lambda+\frac{1}{2})}{2^{2\lambda}.\Gamma(\lambda).n\,!\Gamma(n+2\lambda+1)\Gamma(2m+1+n)}.\exp(-\tfrac{1}{2}z^2)z^nM_{\frac{1}{2}n+\lambda-m,m+\frac{1}{2}n}(z). \qquad (3.3)$$

For $\lambda = \frac{1}{2}$, we get Mitra's results.

4. We now consider the integral I_1,

$$I_1 = \int_0^1 P_n^\lambda(1-2y^2).(1-y^2)^{\lambda-\frac{1}{2}}.y^{\lambda+\frac{3}{2}}.J_{\nu+\frac{1}{2}\lambda-\frac{1}{4}}(yz)\Phi(yz)\,dy\,;\ (\lambda+2\nu>-\tfrac{3}{2}).$$

where

$$\Phi(yz) = (n+\nu+\tfrac{3}{2}\lambda+\tfrac{1}{4})(n+\lambda+\tfrac{1}{2})J_{\nu-\frac{1}{2}\lambda-\frac{3}{4}}(yz) + (n-\nu+\tfrac{3}{2}\lambda+\tfrac{1}{4})(n+2\lambda)J_{\nu-\frac{1}{2}\lambda+\frac{1}{4}}(yz).$$

On using the known formulae for product of two Bessel Functions and the formulae (2.1), we find that

$$I_1 = \left[\frac{(n+\lambda+\frac{1}{2})(\nu-\frac{1}{2}\lambda+\frac{1}{4})(\frac{1}{2}z)^{2\nu-1}}{\Gamma(\nu-\frac{1}{2}\lambda-n+\frac{5}{4})\Gamma(\nu+\frac{3}{2}\lambda+n+\frac{1}{4})}\right.$$

$$\left.+\sum_{r=1}^{\infty}\frac{(-1)^r\Gamma(2\nu+2r).2(\nu-\frac{1}{2}\lambda+\frac{1}{4})\{r(n+\frac{3}{2}\lambda+\frac{1}{4})-(n+2\lambda)\nu\}}{r\,!\Gamma(r+\nu-\frac{1}{2}\lambda-n+\frac{5}{4})\Gamma(r+\nu+\frac{3}{2}\lambda+n+\frac{1}{4})\Gamma(2\nu+r+1)}\times(\tfrac{1}{2}z)^{2\nu+2r-1}\right].\frac{(-1)^n\pi^{\frac{1}{2}}\Gamma(n+2\lambda)}{2^{2\lambda}.n\,!\Gamma(\lambda)}$$

$$= (n+\lambda+\tfrac{1}{2})(\nu-\tfrac{1}{2}\lambda+\tfrac{1}{4})(\tfrac{1}{2}z)^{2\nu-1}.\frac{(-1)^n\pi^{\frac{1}{2}}\Gamma(n+2\lambda)}{2^{2\lambda}.n\,!\Gamma(\lambda)}\left(\sum_{r=0}^{\infty}c_rz^{2r}\right)$$

where

$$(n+\lambda+\tfrac{1}{2})c_r = \frac{(-1)^r\Gamma(2\nu+2r)\,2\{r(n+\frac{3}{2}\lambda+\frac{1}{4})+(n+2\lambda)\nu\}}{r\,!\Gamma(r+\nu-\frac{1}{2}\lambda-n+\frac{5}{4})\Gamma(r+\nu+\frac{3}{2}\lambda+n+\frac{1}{4})\Gamma(2\nu+r+1)}$$

and

$$c_0 = \{\Gamma(\nu-\tfrac{1}{2}\lambda-n+\tfrac{5}{4})\Gamma(\nu+\tfrac{3}{2}\lambda+n+\tfrac{1}{4})\}^{-1}.$$

Consider a set of constants $\{\lambda_r\}$, where

$$\lambda_0 = 1$$

and

$$\lambda_r = \frac{\binom{2\nu+2r+\lambda-\frac{1}{2}}{r}}{\binom{2\nu+2r-1}{r-1}} \cdot \frac{r(n+\lambda+\frac{1}{2})}{2\{r(n+\frac{3}{2}\lambda+\frac{1}{4})+(n+2\lambda)\nu\}} \quad \text{for } r = 1, 2, \ldots$$

Let us denote by

$$\frac{(-1)^n \pi^{\frac{1}{2}} \Gamma(n+2\lambda)}{2^{2\lambda} n!\Gamma(\lambda)} (n+\lambda+\tfrac{1}{2})(\nu-\tfrac{1}{2}\lambda+\tfrac{1}{4})(\tfrac{1}{2}z)^{2\nu-1}\left(\sum_{r=0}^{\infty} c_r \lambda_r z^{2r}\right)$$

the T_λ-transform of the integral I_1. We can then prove that

$$T_\lambda I_1 = (n+\lambda+\tfrac{1}{2})(\nu-\tfrac{1}{2}\lambda+\tfrac{1}{4})(\tfrac{1}{2}z)^{-\lambda-\frac{1}{2}} J_{\nu-\frac{1}{2}\lambda-n+\frac{1}{4}}(z) J_{\nu+\frac{3}{2}\lambda+n-\frac{3}{4}}(z) \tag{4.1}$$

For $\lambda = \frac{1}{2}$, T_λ is the identical transform, and we get the interesting result of Mitra (1936).

5. *Some Applications to the Evaluation of Integrals of Bessel Functions*.

If we put tz for z in (2.2) and multiply both sides by $(tz)^\lambda$ and differentiate with respect to t, then we have

$$\int_0^1 P_n^\lambda(1-2y^2)(1-y^2)^{\lambda-\frac{1}{2}}(2yz)(tyz)^\lambda J_{\lambda-1}(2tyz)dy = \frac{\pi\Gamma(n+2\lambda)}{2^{2\lambda} n!\Gamma(\lambda)} \frac{d}{dt}\{J^2_{n+\lambda}(tz)\}. \tag{5.1}$$

Hence multiplying both sides of (5.1) by $t^\nu(1-t^2)^\mu$ and integrating from zero to one with respect to t and observing that

$$\int_0^1 t^{\lambda+\nu} J_{\lambda-1}(2tyz)(1-t^2)^\mu dt$$

$$= \frac{1}{2}\sum_{r=0}^{\infty} \frac{(-1)^r (yz)^{2r-1+\lambda}}{r!\Gamma(r+\lambda)} \frac{\Gamma(r+\frac{1}{2}\nu+\lambda)\Gamma(\mu+1)}{\Gamma(r+\frac{1}{2}\nu+\mu+\lambda+1)}; \quad (2\lambda+\nu>0,\ \mu>-1),$$

we find that

$$\int_0^1 \frac{d}{dt}\{J^2_{n+\lambda}(tz)\}(1-t^2)^\mu t^\nu dt, \quad (2\lambda+\nu>0,\ \mu>-1),$$

$$= \frac{\Gamma(\mu+1)\Gamma(n+\lambda+\frac{1}{2}\nu)\Gamma(n+\lambda+\frac{1}{2})z^{2n+2\lambda}}{\sqrt{\pi}\,\Gamma(n+\lambda)\Gamma(n+\lambda+\mu+\frac{1}{2}\nu+1)\Gamma(2n+2\lambda+1)}$$

$$\times {}_2F_3(n+\lambda+\tfrac{1}{2}\nu,\ n+\lambda+\tfrac{1}{2};\ n+\lambda,\ n+\lambda+\mu+\tfrac{1}{2}\nu+1,\ 2n+2\lambda+1;\ -z^2) \tag{5.2}$$

As a particular case, let us put $\mu = -\frac{1}{2}$, $\nu = 0$; we have for $\lambda > 0$,

$$\int_0^1 \frac{d}{dt}\{J^2_{n+\lambda}(tz)\}(1-t^2)^{-\frac{1}{2}}dt = J_{2n+2\lambda}(2z) \tag{5.3}$$

Let us put $2z \sin\theta$ for z in (2.4) and write the result as follows:

$$\int_0^1 P_n^\lambda(1-2y^2).(1-y^2)^{\lambda-\frac{1}{2}}.y^{\lambda+\frac{1}{2}} J_{\lambda-\frac{1}{2}}(2yz\sin\theta).$$

$$= \frac{\pi^{\frac{1}{2}}\Gamma(n+2\lambda)}{n!\Gamma(\lambda).2^{\lambda-\frac{1}{2}}} \cdot \frac{J_{2n+2\lambda+1}(2z\sin\theta)+J_{2n+2\lambda-1}(2z\sin\theta)}{2(2n+2\lambda).(2z\sin\theta)^{\lambda-\frac{1}{2}}}. \tag{5.4}$$

Integrating both sides from 0 to $\frac{1}{2}\pi$ and using the wellknown formula

$$J_\nu^2(z) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{2\nu}(2z\sin\theta)d\theta$$

we have

$$\int_0^1 P_n^\lambda(1-2y^2).(1-y^2)^{\lambda-\frac{1}{2}}y^{\lambda+\frac{1}{2}}J^2_{\frac{1}{2}\lambda-\frac{1}{4}}(yz)dy = \frac{\Gamma(n+2\lambda)}{n!\Gamma(\lambda).2^{\lambda-\frac{1}{2}}}\cdot\frac{(2z)^{\frac{1}{2}-\lambda}}{2(2n+2\lambda)}$$

$$\times\left[\frac{z^{2n+2\lambda-1}\Gamma(n+\frac{1}{2}\lambda+\frac{1}{4})}{\Gamma(2n+2\lambda)\Gamma(n+\frac{1}{2}\lambda+\frac{3}{4})}\cdot{}_1F_2(n+\tfrac{1}{2}\lambda+\tfrac{1}{4},\ 2n+2\lambda,\ n+\tfrac{1}{2}\lambda+\tfrac{3}{4};\ -z^2)\right.$$

$$\left.+\frac{z^{2n+2\lambda+1}\Gamma(n+\frac{1}{2}\lambda+\frac{5}{4})}{\Gamma(2n+2\lambda+2)\Gamma(n+\frac{1}{2}\lambda+\frac{7}{4})}\,{}_1F_2(n+\tfrac{1}{2}\lambda+\tfrac{5}{4},\ 2n+2\lambda+2,\ n+\tfrac{1}{2}\lambda+\tfrac{7}{4},\ -z^2)\right]. \quad (5.5)$$

6. We now use the well-known formula

$$(1-2hx+h^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^\lambda(x).h^n$$

to obtain certain integrals involving Bessel functions as a series of Bessel functions.

Put $1-2y^2$ for x ; then

$$\{(1-h)^2+4hy^2\}^{-\lambda} = \sum_{n=0}^{\infty} h^n P_n^\lambda(1-2y^2). \quad (6.1)$$

Multiply both sides of (6.1) by $J_\lambda(2yz)y^\lambda(1-y^2)^{\lambda-\frac{1}{2}}$ and integrate, then we have

$$\int_0^1 J_\lambda(2yz)y^\lambda(1-y^2)^{\lambda-\frac{1}{2}}\{(1-h)^2+4hy^2\}^{-\lambda}dy$$

$$= \sum_{n=0}^{\infty} h^n\int_0^1 P_n^\lambda(1-2y^2)J_\lambda(2yz).y^\lambda.(1-y^2)^{\lambda-\frac{1}{2}}dy.$$

On putting $h = 1$, we have

$$\int_0^1 J_\lambda(2yz)y^{-\lambda}(1-y^2)^{\lambda-\frac{1}{2}}dy = \frac{\pi z^{-\lambda}}{\Gamma(\lambda)}\sum_{n=0}^{\infty}\frac{\Gamma(n+2\lambda)}{n!}J^2_{n+\lambda}(z). \quad (6.2)$$

Similarly putting $h = -1$, we have

$$\int_0^1 J_\lambda(2yz)y^\lambda(1-y^2)^{-\frac{1}{2}}dy = \frac{\pi z^{-\lambda}}{\Gamma(\lambda)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+2\lambda)}{n!}J^2_{n+\lambda}(x) \quad (6.3)$$

Similarly we have from (6.1) and (2.4),

$$\int_0^1 J_{\lambda-\frac{1}{2}}(yz)y^{\frac{1}{2}-\lambda}(1-y^2)^{\lambda-\frac{1}{2}}dy = \frac{\pi^{\frac{1}{2}}}{\Gamma(\lambda)}(2/z)^{\lambda+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(n+2\lambda)}{n!}J_{2n+2\lambda}(z) \quad (6.4)$$

and

$$\int_0^1 J_{\lambda-\frac{1}{2}}(yz)y^{\lambda+\frac{1}{2}}(1-y^2)^{-\frac{1}{2}}dy = \frac{\pi^{\frac{1}{2}}}{\Gamma(\lambda)}(2/z)^{\lambda+\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+2\lambda)}{n!}J_{2n+2\lambda}(z). \quad (6.5)$$

7. We shall now evaluate a few more integrals

On using the Weber-Shafheitlin formula (Watson, 1922)

$$\int_0^\infty J_\mu(bt)\,\frac{J_\nu(a\sqrt{(t^2+z^2)})J_\nu(c\sqrt{(t^2+z^2)})}{(t^2+z^2)^\nu}\,t^{\mu-1}dt$$

$$= \frac{2^{\mu-1}\Gamma(\mu).J_\nu(az)J_\nu(cz)}{b^\mu.z^{2\nu}},\ b > a+c,\ R(2\nu+\tfrac{3}{2}\delta) > R(\mu) > 0.$$

we have from (2.4), after putting $(t^2+z^2)^{\frac{1}{2}}$ for z in (2.4) and after multiplying both sides by

$$J_\mu(bt)\,\frac{J_{\lambda-\frac{1}{2}}(c\sqrt{(t^2+z^2)}).t^{\nu-1}}{(t^2+z^2)^{\lambda-\frac{1}{2}}}$$

the following result:

$$\int_0^\infty J_\mu(bt).\frac{J_{2n+2\lambda}(\sqrt{(t^2+z^2)}).J_{\lambda-\frac{1}{2}}(c\sqrt{(t^2+z^2)})t^{\mu-1}}{(t^2+z^2)^{\frac{1}{2}(8\lambda-\frac{1}{2})}}\,dt$$

$$= \frac{2^{\mu-1}\Gamma(\mu)}{b^\mu.z^{3\lambda-\frac{1}{2}}}\,J_{2n+2\lambda}(yz).J_{\lambda-\frac{1}{2}}(cz) \tag{7.1}$$

for $2\lambda+\frac{3}{2} > \mu > 0,\ \ b > 1+c$.

This result was given by Gegenbauer (1884) when $n = 0$ and $c = 1$. Similarly again putting $\sqrt{(t^2+z^2)}$ for z in (2.4), and multiplying both sides by

$$J_{\mu+1}(bt).t^\mu.(t^2+z^2)^{\frac{1}{2}-\frac{1}{2}\lambda}$$

and on using Sonine's discontinuous integral

$$\int_0^\infty \frac{J_\nu(y\sqrt{(t^2+z^2)})}{(t^2+z^2)^{\frac{1}{2}\nu}}\,J_{\mu+1}(bt).t^\mu dt = \frac{2^\mu\Gamma(\mu+1)}{b^{\mu+1}}.\frac{J_\nu(yz)}{z^\nu};\quad (\nu+1 > \mu > -1),\ (b > y).$$

we have

$$\int_0^\infty \frac{J_{2n+2\lambda}(\sqrt{(t^2+z^2)})}{(t^2+z^2)^\lambda}\,J_{\mu+1}(bt).t^\mu dt = \frac{2^\mu\Gamma(\mu+1)}{b^{\mu+1}z^{2\lambda}}.J_{2n+2\lambda}(z);\ (2\lambda+1 > \mu > -1),\ (b > 1). \tag{7.2}$$

a result which has been given by Gegenbauer (1884) when $n = 0$.

We note here that in all cases term-by-term integration, change in the order of integration and differentiation under the sign of integration are easily justifiable.

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ON RULED SURFACES

By

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1. In this paper we propose to study ruled surfaces by using dualistic vector calculus introduced by E. Study (1903). A lucid exposition of dual vectors applied to line-geometry occurs in the work by W. Blaschke (1924)

We call the rectangular trihedron, determined by the generator, the normal to the ruled surface and the tangent to the ruled surface at the central point of the generator, the Principal Trihedron T. Using this principal trihedron, we find the skewness of distribution, a concept introduced by V. Rangachariar (1945), of the ruled surfaces formed by three edges of the principal trihedron. We also find the condition that the osculating quadrics of a ruled surface be equilateral

2. Using the notion of a dual vector a straight line 'A' is represented by the equation,

$$A = a + \epsilon\bar{a}, \tag{2·1}$$

where 'a' is a unit vector along the generator and the vector '$\bar{a}$' represents the moment of the unit vector along 'A' about the origin of coordinates.

The fundamental relations here are,

$$a.a = 1 \;; \quad a\,\bar{a} = 0. \tag{2·2}$$

As

$$(A.A) = (a.a) + 2\epsilon(a\;\bar{a}) + \epsilon^2(\bar{a}.\bar{a}),$$

the point with the dual coordinates 'A' lies on the unit sphere $(A.A) = 1$,

if

$$\epsilon^2 = 0. \tag{2.3}$$

For determining a ruled surface, we suppose that the line 'A' is the generator of the ruled surface and it is a function of a real parameter 't'. Thus

$$A = A(t) = a(t) + \epsilon\bar{a}(t), \quad \epsilon^2 = 0. \tag{2·4}$$

Let $A_1 = A$, A_2, A_3 be the dual vectors of the three edges of the principal trihedron at the central point of A.

If dashes denote differentiation with respect to 't', then (Blaschke 1946, p. 196-7)

$$A_1' = PA_2 \;;\; A_2' = -PA_1 + QA_3 \;;\; A_3' = -QA_2. \tag{2·5}$$

where P and Q are called the dualistic curvature and dualistic torsion respectively (Biran, 1941, p. 125), and

$$a_1' = pa_2 \;;\; a_2' = -pa_1 + qa_3 \;;\; a_3' = -qa_2 \;; \tag{2·6}$$

$$\bar{a}_1' = \bar{p}a_2 + p\bar{a}_2, \quad \bar{a}_2' = -\bar{p}a_1 + \bar{q}a_3 - p\bar{a}_1 + q\bar{a}_3; \quad \bar{a}_3' = -\bar{q}a_2 - q\bar{a}_2. \tag{2·7}$$

The quantities P and Q are such that

$$P = p + \epsilon\bar{p} = \sqrt{(A_1'^2)}; \quad Q = q + \epsilon\bar{q} = (A_1\, A_1'\, A_1'')/A_1'^2, \tag{2·8}$$

whence

$$\left.\begin{aligned} p &= \sqrt{(a_1'^2)}; \quad \bar{p} = (\bar{a}_1'.a_1')/\sqrt{(a_1'^2)}; \quad q = (a_1\, a_1'\, a_1'')/a_1'^2, \\ \bar{q} &= \{(\bar{a}_1\, a_1'\, a_1'') + (a_1\, \bar{a}_1'\, a_1'') + (a_1\, a_1'\, \bar{a}_1'')\}/a_1'^2 - 2(a_1\, a_1'\, a_1'')(a_1'\, \bar{a}_1')/(a_1'^2)^2 \end{aligned}\right\} \tag{2·9}$$

3. The parameter of distribution of the ruled surface is given by (Blaschke 1924, p. 197)

$$d_1 = \bar{p}/p.$$

The skewness of distribution of the generators of the ruled surface is given by

$$\mu_1 = (a_1\, a_1'\, \bar{a}_1')/(a'^2)^{3/2} = q/p.$$

The object now is to find the skewness of distribution of the normals to this ruled surface at the central points. It is given by

$$\mu_2 = (a_2\, a_2'\, \bar{a}_2')/(a_2'^2)^{3/2},$$

which by virtue of (2.6) becomes

$$\mu_2 = (a_2 \quad -pa_1 + qa_3 \quad -p^2a_2 - q^2a_2 - a_1p' + a_3q')/(p^2+q^2)^{3/2},$$

or

$$\mu_2 = (pq' - p'q)/(p^2+q^2)^{3/2}.$$

μ_2 vanishes if $pq' - p'q = 0$, if $q/p =$ constant, *i.e.* if the skewness of distribution of the original ruled surface is constant. But $\mu_2 = 0$ is the condition that the normals to the ruled surface at the central points are coplanar.

Hence *the normals to a ruled surface at the central points of the generators of the ruled surface are coplanar if the ruled surface has constant skewness of distribution.*

Similarly the skewness of distribution of the ruled surface formed by lines A_3 is given by

$$\mu_3 = (a_3 \quad a_3' \quad \bar{a}_3')/(a_3'^2)^{3/2},$$

or, in consequence of (2.6)

$$\mu_3 = (a_3 \quad -qa_2 \quad pqa_1 - q^2a_3 - q'a_2)/q^3 = pq^2/q^3 = p/q.$$

Hence *the skewness of distribution of the ruled surface formed by the lines A_3 is reciprocal of the skewness of distribution of the original ruled surface.*

The lines A_3 are coplanar if the original ruled surface is such that its spherical representations are minimal lines.

4. We now find the condition that the osculating quadrics of the ruled surface be equilateral.

We know that the condition for this is (Ram Behari, 1946, p. 30)

$$a'.x'' - a''.x' + 2a.x'(a.a'' - a'^2) = 0, \tag{4.1}$$

where x is the point on the base curve through which the generator passes,

Let x be the vector of which the origin is the fixed point O and the extremity is the origin M of the principal trihedron. T. We then have (Blaschke, 1924, p. 197)

$$x' = \bar{q}a_1 + \bar{p}a_3. \tag{4.2}$$

Taking the length of the arc of the line of striction of the ruled surface as a parameter, we have

$$x'^2 = 1,$$

and hence

$$\bar{p}^2 + \bar{q}^2 = 1.$$

Use of (4.2) and (2.6) in (4.1) yields

$$pa_2(\bar{q}pa_2 - \bar{p}qa_2 + a_1\bar{q}' + a_3\bar{p}') - (\bar{q}a_1 + \bar{p}a_3)(-p^2a_1 + pqa_3 + a_2p')$$
$$+ 2a_1.(\bar{q}a_1 + \bar{p}a_3)[a_1.(-p^2a_1 + pqa_3 + a_2p') - p^2] = 0,$$

or

$$p(p\bar{q} + q\bar{p}) = 0,$$

i.e., either $p = 0$, which means *that the surfaces whose spherical representations are minimal lines have equilateral osculating quadrics*,

or

$$p\bar{q} + q\bar{p} = 0.$$

The ruled surfaces formed by the normals at the central points, have equilateral osculating quadrics if

$$a'_2.x'' - a''_2.x' + 2a_2.x'(a_2.a''_2 - a'^2_2) = 0.$$

Using (2.6) and (4.2) in this equation we get,

$$(-pa_1 + qa_3)(\bar{q}pa_2 - \bar{p}qa_2 + a_1\bar{q}' + a_3\bar{p}') - (-p^2a_2 - q^2a_2 - a_1p' + a_3q')(\bar{q}a_1 + \bar{p}a_3)$$
$$+ 2a_2(\bar{q}a_1 + \bar{p}a_3)[a_2.(-p^2a_2 - q^2a_2 - a_1p' + a_3q') - p^2 - q^2] = 0$$

or

$$p'\bar{q} + q\bar{p}' = p\bar{q}' + q'\bar{p}.$$

The ruled surfaces formed by the lines A_3 have equilateral osculating quadrics if

$$a'_3.x'' - a''_3.x' + 2a_3.x'(a_3.a''_3 - a'^2_3) = 0.$$

or

$$-qa_2(\bar{q}pa_2 - \bar{p}qa_2 + a_1\bar{q}' + a_3\bar{p}') - (qpa_1 - q^2a_3 - a_2q').(\bar{q}a_1 + \bar{p}a_3)$$
$$+ 2a_3.(\bar{q}a_1 + \bar{p}a_3)[a_3.(qpa_1 - q^2a_3 - a_2q') - q^2] = 0,$$

or

$$q(p\bar{q} + q\bar{p}) = 0.$$

Therefore the ruled surfaces formed by the lines A_3 have equilateral osculating quadrics if either $p\bar{q} + q\bar{p} = 0$ or $q = 0$. The second condition is equivalent to the condition that the generators of the original ruled surface are coplanar. The first is the condition sufficient to make both the surfaces formed by the lines A_1 and A_3 equilateral.

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ON SOME OPERATIONAL AND OTHER RELATIONS INVOLVING TSCHEBYSCHEFF'S AND LAGUERRE POLYNOMIALS

By

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Introduction. The paper deals with the derivation of two formulae in operational calculus connecting Tschebyscheff's polynomial $T_n(z)$ respectively with the parabolic cylinder function $D_n(z)$ of integral order n and the product of two Laguerre polynomials. A few results connecting Laguerre polynomials with the parabolic cylinder functions as also Hermite polynomials have been obtained. Hankel Transforms of orders 0 and 1 and some infinite integrals obtained as a direct consequence, have been studied. In what follows, we have defined $T_n(z)$ by

$$T_n(z) = (1/2^{n-1}) \cos (n \cos^{-1}z). \quad [\text{Hobson}]$$

The results obtained in this paper are believed to be new.

1. From Mehler's formula involving the product of two parabolic cylinder functions, *viz*,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!\sqrt{(\frac{1}{2}\pi)}}.D_n(x)D_n(y) = \frac{1}{\sqrt{\{\frac{1}{2}\pi(1-t^2)\}}} \exp. \left\{\frac{4xyt-(x^2+y^2)(1+t^2)}{4(1-t^2)}\right\}$$

we have, on substitution,

$$x+y = 2u \quad \text{and} \quad x-y = 2v$$

and simplification, the relation

$$(1-t^2)^{\frac{1}{2}} \sum_{r=0}^{\infty} \frac{t^r}{r!} D_r(u+v)D_r(u-v) = \exp\left\{\frac{u^2}{2}\cdot\frac{t-1}{t+1}\right\} \exp\left\{\frac{v^2}{2}\cdot\frac{t+1}{t-1}\right\}. \tag{1.1}$$

In virtue of

$$\sum_{r=0}^{\infty} t^r k_{2r}(x) = \exp\{x(t-1)/(t+1)\} \tag{1.2}$$

where $k_{2n}(x)$ is Bateman's k-function, the equality (1.1) takes the form

$$\sum_{r=0}^{\infty} \frac{\Gamma(r-\frac{1}{2})}{r!\Gamma(-\frac{1}{2})}.t^{2r}. \sum_{r=0}^{\infty} \frac{t^r}{r!} D_r(u+v)D_r(u-v) = \sum_{r=0}^{\infty} t^r k_{2r}(\tfrac{1}{2}u^2) \sum_{r=0}^{\infty} (-1)^r t^r k_{2r}(\tfrac{1}{2}v^2). \tag{1.3}$$

It may be noted that the domain of convergence in each of (1.1), (1.2) and so in (1.3) is $|t| < 1$.

Equating coefficients of t^{2r+1} and t^{2r} from both sides of (1.3) we obtain

$$\frac{1}{2\sqrt{\pi}} \sum_{s=0}^{r} \frac{\Gamma(r-s-\frac{1}{2})}{(r-s)!(2s+1)!} D_{2s+1}(u+v)D_{2s+1}(u-v) = \sum_{s=0}^{2r+1} (-1)^{s+1} k_{2s}(\tfrac{1}{2}v^2) k_{2(2r+1-s)}(\tfrac{1}{2}u^2) \tag{1.4}$$

and

$$\frac{1}{2\sqrt{\pi}} \sum_{s=0}^{r} \frac{\Gamma(r-s-\frac{1}{2})}{(r-s)!(2s)!} D_{2s}(u+v)D_{2s}(u-v)$$

$$= \sum_{s=0}^{2r} (-1)^{s+1} k_{2s}(\tfrac{1}{2}v^2) k_{2(2r-s)} (\tfrac{1}{2}u^2) \tag{1.5}$$

where we have utilised the relation

$$\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}.$$

(1.4) and (1.5) therefore connect parabolic cylinder functions with Bateman's k-function. The relation

$$L_n(x) - nL_{n-1}(x) = (-1)^n n!\, e^{\frac{1}{2}x} k_{2n}(\tfrac{1}{2}x)$$

being kept in view, it is evidently possible to obtain from (1.4) and (1.5) a connection of Laguerre polynomials with the products of Parabolic cylinder functions.

2. Proceeding next, as in the foregoing article, with the generating function (Titchmarsh, 1948)

$$\sum_{n=0}^{\infty} \frac{t^n e^{-x^2}\{H_n(x)\}^2}{2^n n!\sqrt{\pi}} = \frac{1}{\sqrt{\{\pi(1-t^2)\}}} \exp\left(-x^2 \frac{1-t}{1+t}\right), \quad |t| < 1$$

where $H_n(x)$ is the Hermite polynomial, we obtain

$$\frac{1}{\Gamma(-\frac{1}{2})} \sum_{s=0}^{r} \frac{\Gamma(r-s-\frac{1}{2})}{2^{2s+1}(r-s)!(2s+1)!} \{H_{2s+1}(x)\}^2 = \exp(x^2).k_{2(2r+1)}(x^2). \tag{2.1}$$

Whence putting $x^2 = \frac{1}{2}z$ and making use of the operational representation

$$\exp(\tfrac{1}{2}z).k_{2n}(\tfrac{1}{2}z) \doteqdot p^{-1}(p^{-1}-1)^{n-1}$$

we have

$$\frac{1}{\Gamma(-\frac{1}{2})} \sum_{s=0}^{r} \frac{\Gamma(r-s-\frac{1}{2})}{2^{2s+1}(r-s)!(2s+1)!} \{H_{2s+1}(\sqrt{(z/2)}\}^2 \doteqdot p^{-1}(p^{-1}-1)^{2r}. \tag{2.2}$$

Interpretation of the right hand side by

$$\frac{L_n(z)}{n!} \doteqdot \left(\frac{p-1}{p}\right)^n$$

leads at once to the relation

$$\frac{(2r+1)!}{2\sqrt{\pi}} \sum_{s=0}^{r} \frac{\Gamma(r-s-\frac{1}{2})}{2^{2s+1}(r-s)!(2s+1)!} \{H_{2s+1}(\sqrt{(z/2)}\}^2 = L_{2r+1}(z) - (2r+1)L_{2r}(z) \tag{2.3}$$

which may also be otherwise derived. Similarly

$$\frac{(2r)!}{2\sqrt{\pi}} \sum_{s=0}^{r} \frac{\Gamma(r-s-\frac{1}{2})}{2^{2s}(r-s)!(2s)!} \{H_{2s}(\sqrt{(z/2)}\}^2 = 2rL_{2r-1}(z) - L_{2r}(z) \tag{2.4}$$

From (2.3) and (2.4) making use of the result,

$$H_n(x) = 2^{\frac{1}{2}n} \exp(\tfrac{1}{2}x^2).D_n(x\sqrt{2}), \tag{2.5}$$

$$\frac{(2r+1)!}{2\sqrt{\pi}} \sum_{s=0}^{r} \frac{\Gamma(r-s-\frac{1}{2})}{(r-s)!(2s+1)!} \{D_{2s+1}(z)\}^2 = \exp(-\tfrac{1}{2}z^2).[L_{2r+1}(z^2) - (2r+1)L_{2r}(z^2)] \tag{2.6}$$

and

$$\frac{(2r)!}{2\sqrt{\pi}} \sum_{s=0}^{r} \frac{\Gamma(r-s-\frac{1}{2})}{(r-s)!(2s)!} \{D_{2s}(z)\}^2 = \exp(-\tfrac{1}{2}z^2).[2rL_{2r-1}(z^2) - L_{2r}(z^2)]. \tag{2.7}$$

We conclude from (2.6) that the function

$$\exp(-\tfrac{1}{2}z^2)z^{-\frac{1}{2}}[L_{2r+1}(z^2) - (2r+1)L_{2r}(z^2)]$$

*and so $z^{-\frac{1}{2}}k_{2(2r+1)}(\frac{1}{2}z^2)$, is self reciprocal in the Hankel's Transform of order 1, since $z^{-\frac{1}{2}}\{D_{2s+1}(z)\}^2$ is so.**

3. Representing the L H. S. of (2.2) by $F(z)$, we readily obtain

$$\exp(-z).F(z) \doteqdot p^{2r+1}/(p+1)^{2r+2} = f(p), \text{ say,} \tag{3.1}$$

which, by virtue of the relation

$$p^{-m+1}f(1/p) \doteqdot \int_0^\infty (z/s)^{\frac{1}{2}m}J_m(2\sqrt{(zs)})h(s)ds$$

when $f(p) \doteqdot h(z)$, leads to

$$\int_0^\infty (z/s)^{\frac{1}{2}}J_1(2\sqrt{(zs)})\exp(-s)F(s)ds = \frac{\exp(-z)z^{2r+1}}{\Gamma(2r+2)}, \tag{3.2}$$

since

$$\frac{p}{(p+1)^{2r+2}} \doteqdot \frac{\exp(-z)z^{2r+1}}{\Gamma(2r+2)}.$$

In terms of k-function (3.2) reduces to

$$\int_0^\infty (z/s)^{\frac{1}{2}}J_1(2\sqrt{(zs)})\exp(-\tfrac{1}{2}s)k_{2(2r+1)}(\tfrac{1}{2}s)ds = \frac{\exp(-z).z^{2r+1}}{\Gamma(2r+2)}. \tag{3.3}$$

Again, from the operational representation

$$\left(-p\frac{d}{dp}\right)^n\phi(p) \doteqdot \left(z\frac{d}{dz}\right)^n h(z), \quad n>0$$

we have

$$zF'(z) \doteqdot -p\frac{d}{dp}\{p^{-1}(p^{-1}-1)^{2r}\} = (2r+1)p^{-1}(p^{-1}-1)^{2r}+2rp^{-1}(p^{-1}-1)^{2r-1}.$$

Hence proceeding similarly as in (3.2) we derive

$$\int_0^\infty \sqrt{(zs)}J_1(2\sqrt{(zs)})\exp(-s)F'(s)ds = \frac{\exp(-z)z^{2r}}{\Gamma(2r+1)}(z+2r) \tag{3.4}$$

Further, with $f(p)$ having the same meaning as in (3.1) we have

$$f(1/p) = p/(1+p)^{2r+2}$$

So that, by the operational relation

$$pf(1/p) \doteqdot \int_0^\infty J_0(2\sqrt{(zs)})h(s)ds$$

we have

$$\int_0^\infty J_0(2\sqrt{(zs)})\exp(-s)F(s)ds = \frac{\exp(-z)z^{2r}}{\Gamma(2r+2)}[2r+1-z]. \tag{3.5}$$

Proceeding similarly with

$$\psi(z) = \frac{1}{\Gamma(-\frac{1}{2})}\sum_{s=0}^{r}\frac{\Gamma(r-s-\frac{1}{2})}{2^{2s}(r-s)!(2s)!}\{H_{2s}(\sqrt{(z/2)})\}^2$$

---

* Vide Mitra S. C. (1936) and a remark by Watson (1936). Also Titchmarsh (1948, p. 262). It may be remarked that similar conclusion cannot be made for the function defined by the R. S. of (2.7).

we can deduce integrals of the types (3.2)—(3.5) for the function $\psi(z)$, consequently

$$\int_0^\infty (z/s)^{\frac{1}{2}} J_1(2\sqrt{(zs)}) \exp(-s)\psi(s)ds = \frac{\exp(-z).z^{2r}}{\Gamma(2r+1)}. \tag{3.6}$$

Writing the relation (2.1) in the form

$$\frac{1}{\Gamma(-\frac{1}{2})}\sum_{s=0}^{r}\frac{\Gamma(r-s-\frac{1}{2})}{(r-s)!(2s+1)!}\{D_{2s+1}(z)\}^2 = k_{2(2r+1)}(\tfrac{1}{2}z^2) \tag{3.7}$$

by (2.5), the following infinite integrals involving k-functions may be immediately derived.

Since

$$\int_{-\infty}^{\infty}\{D_n(z)\}^2 dz = (2\pi)^{\frac{1}{2}} n! \quad (M.\ A., \text{ p. } 351).$$

We have

$$\int_{-\infty}^{\infty} k_{2(2r+1)}(\tfrac{1}{2}z^2)dz = -\frac{1}{\sqrt{2}}\sum_{s=0}^{r}\frac{\Gamma(r-s-\frac{1}{2})}{(r-s)!} \tag{3.8}$$

combining Howell's formula (Howell, 1938) *viz.*

$$\int_0^\infty \exp(-sx).x^{-\frac{1}{2}} D_n(2\sqrt{(\lambda x)})D_n(2\sqrt{(\mu x)})dx = n!\sqrt{\pi}\,\frac{(\lambda+\mu-s)^{\frac{1}{2}n}}{(\lambda+\mu+s)^{\frac{1}{2}(n+1)}}\,P_n\left\{\sqrt{\left(\frac{4\lambda\mu}{(\lambda+\mu)^2-s^2}\right)}\right\}$$

with (3.7), we have, after slight simplification,

$$\int_0^\infty \exp(-x\cos 2\theta).x^{-\frac{1}{2}} k_{2(2r+1)}(x)dx$$

$$= -\frac{1}{2\sqrt{2}}\sum_{s=0}^{r}\frac{\Gamma(r-s-\frac{1}{2})}{(r-s)!}(\tan\theta)^{2s+1}\sec\theta.P_{2s+1}(\operatorname{cosec} 2\theta). \tag{3.9}$$

Further, since (Howell, 1938).

$$\int_{-\infty}^{\infty} \exp(2aix).D_n(x\sqrt{2})D_m(x\sqrt{2})dx = \sqrt{\pi}\, n!(ia\sqrt{2})^{m-n}\exp(-a^2)L_n^{m-n}(2a^2),$$

where m and n are positive integers and $L_n^a(x)$ is the Laguerre polynomial, we have from (3.7), on the assumption that a and x are real, the integral

$$\int_{-\infty}^{\infty} k_{2(2r+1)}(x^2)\cos(2ax)\,dx = -\tfrac{1}{2}\exp(-a^2).\sum_{s=0}^{r}\frac{\Gamma(r-s-\frac{1}{2})}{(r-s)!(2s+1)!}L_{2s+1}(2a^2). \tag{3.10}$$

Similar integrals may also be obtained for $k_{1r}(x)$. Also by putting $t = \exp(2i\theta)$ in (1.2) we derive the relations*

$$\sum_{r=0}^{\infty} k_{2r}(x)\cos 2r\theta = \cos(x\tan\theta) \tag{3.11}$$

$$\sum_{r=0}^{\infty} k_{2r}(x)\sin 2r\theta = \sin(x\tan\theta), \quad (0 < \theta < \pi/2). \tag{3.12}$$

4. Writing the relation (Bagchi & Chakravarty, 1950)

$$2^r T_r(z) = \sum_{m+n=r}\sum P_m(z)P_n(z) - \sum_{m+n=r-2}\sum P_m(z)P_n(z)$$

* These were obtained by Shastri (1935) by a different method.

in the form

$$2^rT_r(z) = \sum_{m=0}^{r} P_m(z)P_{r-m}(z) - \sum_{m=0}^{r-2} P_m(z)P_{r-m-2}(z) \tag{4.1}$$

and expanding $P_m(z)P_n(z)$ by Adam's formula (*M. A.*, p. 331) we have

$$2^rT_r(z) = \sum_{m=0}^{r}\sum_{s=0}^{r-m} \frac{A_{r-m-s}A_sA_{m-s}}{A_{r-s}}\left(\frac{2r-4s+1}{2r-2s+1}\right)P_{r-2s}(z)$$

$$-\sum_{m=0}^{r-2}\sum_{s=0}^{r-m-2} \frac{A_{r-m-s-2}A_sA_{m-s}}{A_{r-s-2}}\left(\frac{2r-4s-3}{2r-2s-3}\right)P_{r-2s-2}(z) \tag{4.2}$$

where

$$A_s = \frac{1.3.5.\ldots(2s-1)}{s\,!}.$$

The relation (4.2) leads to the operational representation

$$2^rT_r(1-2\lambda^2/p^2) \doteqdot \sum_{m=0}^{r}\sum_{s=0}^{r-m} \frac{A_{r-m-s}A_sA_{m-s}}{A_{r-s}}\left(\frac{2r-4s+1}{2r-2s+1}\right)L_{r-2s}(\lambda x)\,L_{r-2s}(-\lambda x)$$

$$-\sum_{m=0}^{r-2}\sum_{s=0}^{r-m-2} \frac{A_{r-m-s-2}A_sA_{m-s}}{A_{r-s-2}}\left(\frac{2r-4s-3}{2r-2s-3}\right)L_{r-2s-2}(\lambda x)L_{r-2s-2}(-\lambda x). \tag{4.3}$$

by a result due to Howell (1937).

The relation (4.3) connects Tschebyscheff's polynomial operationally with the product of Laguerre polynomials.

5. From the generating functions of Tschebyscheff's polynomial $T_n(z)$ and Gegenbauer function $C_n^{\nu}(z)$ given respectively by

$$\sum_{n=0}^{\infty} 2^nt^nT_n(z) = (1-t^2)/(1-2zt+t^2) \quad \text{[Hobson p. 87]}$$

and

$$\sum_{n=0}^{\infty} t^nC_n^{\nu}(z) = (1-2zt+t^2)^{-\nu} \quad [M.\ A.,\ \text{p. } 335]$$

it follows that

$$2^nT_n(z) = C_n^1(z) - C_{n-2}^1(z). \tag{5.1}$$

In the operational relation (Howell, 1937), *viz*

$$x^{2\alpha}L_n^{\alpha}(\lambda x)L_n^{\alpha}(\mu x) \doteqdot \frac{\Gamma(n+\alpha+1)\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+1)}{n\,!\sqrt{\pi}\,2^{-2\alpha}}\cdot p$$

$$\times\sum_{r=0}^{\infty} \frac{(-1)^r a^{-2\alpha-1}}{r\,!\,\Gamma(\alpha-r+1)}\left(\frac{c}{a}\right)^{n-r}\cdot C_{n+r}^{\alpha+\frac{1}{2}}\left(\frac{b^2}{ac}\right).$$

where

$$a^2 = p^2,\ b^2 = p^2-(\lambda+\mu)p+2\lambda\mu$$

and

$$c^2 = \{p-\tfrac{1}{2}(\lambda+\mu)\}^2,$$

put

$$\alpha = \tfrac{1}{2},\ \lambda = -\mu.$$

Then since

$$L_n^{\frac{1}{2}}(x) = \frac{(-)^n}{2^{n+\frac{1}{2}}n\,!}\,x^{-\frac{1}{2}}\exp(\tfrac{1}{2}x)D_{2n+1}(\sqrt{(2x)}),$$

where n is a positive integer, we derive on simplification,

$$-\frac{i}{2^{2n+1}n\,!\Gamma(n+\frac{3}{2})}\cdot\frac{1}{\lambda}D_{2n+1}(\sqrt{(2\lambda x)}).D_{2n+1}(i\sqrt{(2\lambda x)})$$

$$\doteqdot\frac{1}{p}\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\Gamma(\frac{3}{2}-r)}C^1_{n+r}\left(\frac{p^2-2\lambda^2}{p^2}\right). \tag{5.2}$$

The series on the right of (5.2) may easily be seen to be convergent if we take note of the fact that

$$C^1_n(x)=\frac{\sin(n+1)\theta}{\sin\theta}\text{ where }\theta=\cos^{-1}x.$$

Similarly

$$-\frac{i}{2^{2n-3}(n-2)\,!\Gamma(n-\frac{1}{2})}\cdot\frac{1}{\lambda}D_{2n-3}(\sqrt{(2\lambda x)})D_{2n-3}(i\sqrt{(2\lambda x)})$$

$$\doteqdot\frac{1}{p}\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\Gamma(\frac{3}{2}-r)}C^1_{n+r-2}\left(\frac{p^2-2\lambda^2}{p^2}\right) \tag{5.3}$$

Taking (5.2), (5.3) and (5.1) together, we obtain

$$\frac{\lambda}{p}\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\Gamma(\frac{3}{2}-r)}2^{n+r}T_{n+r}\left(\frac{p^2-2\lambda^2}{p^2}\right)\doteqdot\frac{i}{2^{2n-3}(n-2)\,!\Gamma(n-\frac{1}{2})}D_{2n-3}(\sqrt{(2\lambda x)})D_{2n-3}(i\sqrt{(2\lambda x)})$$

$$-\frac{i}{2^{2n+1}n\,!\Gamma(n+\frac{3}{2})}D_{2n+1}(\sqrt{(2\lambda x)})D_{2n+1}(i\sqrt{(2\lambda x)}). \tag{5.4}$$

That the infinite series on the left hand side of (5.4) is convergent may easily be seen from the fact that $T_n(z)\leq 1/2^{n-1}$ for positive integral values of n and $\Gamma(n+1)=\exp(-n)n^n(2\pi n)^{\frac{1}{2}}$, when n is very large, it being implied that in all our discussions, z is real and $|z|<1$.

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ON AN ALGEBRAIC SYSTEM GENERATED BY A SINGLE ELEMENT AND ITS APPLICATION IN RIEMANNIAN GEOMETRY—III

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1. This paper is a continuation of a previous paper (Sen, 1950) in which a non-associative algebraic system was obtained as an extension, satisfying certain conditions, of a finite cyclic sequence of elements. The sequence, on the other hand, was constructed out of an arbitrary abstract element and two distinct operations possessing specific properties. As the algebraic system was seen to have interesting properties and applications in Riemannian geometry when the number of distinct terms of the sequence was greater than four but divisible by four, it seems worthwhile to study the properties of such sequences somewhat closely. It is seen in this paper that an arbitrary sequence of elements of this type may generate other sequences which have interesting group properties The non-associative algebraic systems obtained as extensions of these sequences in the manner stated above form a group and therefore an associative system. These algebraic systems in their applications in Riemannian geometry give us systems of affine connections in each of which the Christoffel symbol is identified in the same way as it was identified is the paper referred to above.

Let us start with the construction of sequences of functions of integers having the proposed property : Let $p = 4n$ be a given number arbitrarily chosen, where $n > 1$ is an integer, and let j run over the successive positive integers $1, 2, \ldots, p$, mod. p. Of course $p = 0$, (mod p) ; but there is no harm to retain p insted of 0. Further, let $\psi(j)$ and $\chi(j)$ be two functions of j defined by the following properties :

(1) The functions are linear and distinct from one another and distinct from j itself.

(2) The functions take only the cyclic permutations of $1, 2, \ldots, p$, mod. p, when the argument j takes its prescribed values.

(3) The functions satisfy

$$\psi(\psi(j)) = \chi(\chi(j)) = j \tag{1.1}$$

The functions $\psi(j)$ and $\chi(j)$ can evidently be regarded as linear transformations of j and, as is generally the case, their product is non-commutative but associative. Writing for the moment ψ, $\psi\chi$, $\psi\chi\psi$, ... for $\psi(j)$, $\psi(\chi(j))$, $\psi(\chi(\psi(j)))$, ... respectively, we have

$$\psi\chi \neq \chi\psi \text{ but } (\psi\chi)\psi = \psi(\chi\psi)$$

(4) The following sequence of linear functions of j

$$j,\ \psi,\ \psi\chi,\ \psi\chi\psi,\ \psi\chi\psi\chi,\ \ldots \tag{1.2}$$

is a finite sequence with the first p terms distinct.(†)

(†) Property (3) implies that j, $\psi(j)$ as well as j, $\chi(j)$ are groups of order 2 ; and the properties (3) together imply that (1 2) is a cyclic sequence whose terms form a group of order p.

We now investigate the consequences of the above four properties. Let $\psi(j) = k+qj$, where the coefficients k and q are obviously integers. It follows from (2) that $q \equiv \pm 1$ (mod. p). So $\psi(j) = k \pm j$. When the $+$ sign is taken, $k \neq p$ (mod. p), by (1) ; but $2k = p$ (mod. p), by (3) Therefore, we have $k = pr/2$, where r is an odd integer. On the other hand, when $-$ sign is taken, two cases arise according as $k = p$ or $k \neq p$ (mod. p). In either of these cases $\psi(j)$ satisfies (3) automatically. Thus there are three possibilities of $\psi(j)$:

$$(i) \quad \psi(j) = pr/2+j, \qquad (ii) \quad \psi(j) = -j, \qquad (iii) \quad \psi(j) = l-j,$$

where r is an odd integer and l is an integer $\neq p$ (mod. p). Similarly for the function $\chi(j)$.

In order to apply property (4) we have to take pairs of the above three functions and examine the nine different cases that may arise :

(*a*) $\psi(j) = pr/2+j$, $\chi(j) = ps/2+j$. The sequence (1.2) is, in this case,

$$j, \quad pr/2+j, \quad p(r+s)/2+j = j \text{ (mod. } p), \; \ldots .$$

Hence the sequence is cyclic with 2 terms contrary to (4).

(*b*) $\psi(j) = pr/2+j$, $\chi(j) = -j$. The sequence is

$$j, \; pr/2+j, \; pr/2-j, \; -j, \; j, \; \ldots .$$

Here the sequence is cyclic with 4 terms contrary to (4).

(*c*) $\psi(j) = pr/2+j$, $\chi(j) = l-j$. The sequence is

$$j, \; pr/2+j, \; pr/2+l-j, \; l-j, \; j, \; \ldots .$$

This case is similar to (*b*).

(*d*) $\psi(j) = -j$, $\chi(j) = pr/2+j$.

Here the sequence is the same as that under (*b*) but with opposite sense.

(*e*) $\psi(j) = \chi(j) = -j$.

The case is contrary to (1).

(*f*) $\psi(j) = -j$, $\chi(j) = l-j$. The sequence is

$$j, \; -j, \; -l+j, \; -l-j, \; -2l+j, \; -2l-j, \; -3l+j, \; -3l-j, \; \ldots$$

As $p = 4n$, $n > 1$, the $(p+1)$th term of the sequence $= -2nl+j = j$ (mod. p) for all n if and only if l is even. So let $|l| = 2r$.

Now $(2t+1)$th term $= \pm 2tr+j$. If follows from (4) that for all $t < p/2$, $2tr \neq p$ (mod. p), *i.e.*, r and $p/2$ are relatively prime, *i.e.*, their $h.\,c\,f.\,(r, p/2) = 1$. Accordingly this case satisfies property (4) only when

$$\psi(j) = -j, \quad \chi(j) = l-j, \quad (|l|, p) = 2.$$

(*g*) $\psi(j) = l-j$, $\chi(j) = pr/2+j$

The sequence is the same as that under (*c*) except for the sense.

(*h*) $\psi(j) = l-j$, $\chi(j) = -j$

This case is the same as (*f*) except for the sense of the sequence.

(*i*) $\psi(j) = l-j$, $\chi(j) = m-j$, $l-m \neq p$ (mod. p). The sequence is

$$j, \; l-j, \; l-m+j, \; 2l-m-j, \; 2l-2m+j, \; 3l-2m-j, \; 3l-3m+j, \; \ldots$$

As $p = 4n$, $n > 1$, the $(p+1)$th term $= 2n(l-m)+j = j$ (mod. p) for all n if and only if $l-m$ is even. So let $|l-m| = 2r$. As in (*f*), this case is possible only when $(|l-m|, p) = 2$. It may be noticed that this case includes (*f*) as a particular case when either l or m is equal to p (mod. p).

All possible cases are now exhausted. We may therefore state the following result: The pair of functions $\psi(j)$, $\chi(j)$ defined by the properties (1) to (4) are of the type

$$\psi(j) = l-j, \quad \chi(j) = m-j, \quad (|l-m|, p) = 2 \tag{1.3}$$

2. For the sake of simplicity we suppose that $l-m = 2$ and (1.3) written as

$$\psi(j) = l-j, \; \chi(j) = l-2-j, \quad j, l = 1, \ldots, p \text{ (mod. } p) \tag{2.1}$$

The sequence (1.2) is now the following:

$j,\ l-j,\ 2+j,\ l+2-j,\ 4+j,\ l+4-j,\ 6+j, \ldots,\ p/2+j,\ l+p/2-j, \ldots,\ l-2-j.$

Let the successive terms of this sequence be denoted by ${}^l\theta_k(j)$, $k = 1, \ldots, p$. The sequence is given by

$${}^l\theta_k(j) = \begin{cases} k-1+j, & \text{if } k \text{ is odd} \\ l+k-2-j, & \text{if } k \text{ is even} \end{cases} \qquad k = 1, \ldots, p. \tag{2.3}$$

For given values of l and k arbitrarily chosen, each term ${}^l\theta_k(j)$ is a cyclic sequence with p distinct terms, for $j = 1, \ldots, p$, and it satisfies

$${}^l\theta_{k+p/2}(j) = {}^l\theta_k(j+p/2) = {}^l\theta_k(j)+p/2 \tag{2.3}$$

The sequence (2.2) gives rise to p sequences for $l = 1, \ldots, p$. With regard to these sequences we observe the following properties:

(1) As ${}^l\theta_k(j)$ is independent of or dependent on l according as k is odd or even, two cases arise:

(*i*) For a given value r, the terms ${}^l\theta_{2r+1}(j)$ are equal to one another whatever value may have.

(*ii*) With respeet to the terms ${}^l\theta_{2r}(j)$, two sub-cases arise according as l is odd or even:

(*a*) For a given s, the terms ${}^{2s-2r+1}\theta_{2r}(j)$ are equal to one another whatever value r may have.

(*b*) For a given s, the terms ${}^{2s-2r+2}\theta_{2r}(j)$ are equal to one another whatever value r may haxe.

It therefore follows that of the p^2 terms ${}^l\theta_k(j)$, only $p+p/2$ are distinct.

(2) The p sequences may be devided into two classes, according as l is odd or even, having the following property: Two sequences of the same class can be transformed into one another by a suitable permutation of the terms; but this is not true of two sequences belonging to different classes.

Consider any one of the p sequences (2.2) for a given value of l chosen arbitrarily, and write θ_k for ${}^l\theta_k(j)$. The multiplication-table of the group formed by the terms of this sequence is given below for future reference:

$$
\begin{array}{c|ccccccccc}
 & \theta_1 & \theta_2 & \theta_3 & \ldots & \theta_{p/2} & \theta_{p/2+1} & \ldots & \theta_{p-1} & \theta_p \\
\hline
\theta_1 & \theta_1 & \theta_2 & \theta_3 & \ldots & \theta_{p/2} & \theta_{p/2+1} & \ldots & \theta_{p-1} & \theta_p \\
\theta_2 & \theta_2 & \theta_1 & \theta_p & . & \theta_{p/2+3} & \theta_{p/2+2} & . & \theta_4 & \theta_3 \\
\theta_3 & \theta_3 & \theta_4 & \theta_5 & . & \theta_{p/2+2} & \theta_{p/2+3} & \ldots & \theta_1 & \theta_2 \\
\ldots & \ldots & \ldots & . & & . & . & . & . & \\
\theta_{p/2} & \theta_{p/2} & \theta_{p/2-1} & \theta_{p/2-2} & \ldots & \theta_1 & \theta_p & \ldots & \theta_{p/2+2} & \theta_{p/2+1} \\
\theta_{p/2+1} & \theta_{p/2+1} & \theta_{p/2+2} & \theta_{p/2+3} & & \theta_p & \theta_1 & & \theta_{p/2-1} & \theta_{p/2} \\
\ldots & & \ldots & \ldots & \ldots & & \ldots & \ldots & \ldots & \\
\theta_{p-1} & \theta_{p-1} & \theta_p & \theta_1 & . & \theta_{p/2-2} & \theta_{p/2-1} & & \theta_{p-3} & \theta_{p-2} \\
\theta_p & \theta_p & \theta_{p-1} & \theta_{p-2} & . & \theta_{p/2+1} & \theta_{p/2} & . & \theta_2 & \theta_1
\end{array}
\tag{2.4}
$$

This group has subgroups whose properties can be studied by the help of well-known results of the theory of finite groups ; but we shall not discuss the subgroups here. We shall merely point out the following properties of the group :

(3) Of the p number of groups (2.4) given by $l = 1, \ldots, p$, only two are distinct by virtue of (2) above, one given by odd values of l and the other by even. And these two groups have $p/2$ elements in common by virtue of (1) (i).

We shall find it convenient to adopt the following notation : Since the θ_k's form a group, given two positive integers m and k, there exists a unique positive integer s such that

$$\theta_m \theta_k = \theta_s, \qquad m, k, s = 1, \ldots, p \ (\text{mod. } p)$$

Let this relation between the three integers be denoted by

$$(m, k) = s$$

Obviously (m, k) is an ordered pair. It follows from (2.4) that

$$
\begin{gathered}
(k, k) = 2k-1, \quad (k, 2-k) = 1, \text{ if } k \text{ is odd} \\
(k, k) = 1, \text{ if } k \text{ is even}
\end{gathered}
\tag{2.5}
$$

Also it follows from (2.3) that that

$$(m+p/2, k) = (m, k+p/2) = (m, k)+p/2 \tag{2.6}$$

Further, since a group satisfies the associative law, we have

$$(n, (m, k)) = ((n, m), k) \tag{2.7}$$

The group (2.4) can be put in a $1-1$ correspondence with a permutation group as follows. We have seen that each term of the cyclic sequence (2.2), namely,

$$\theta_1(j), \ \theta_2(j), \ \ldots, \theta_p(j)$$

is a cyclic sequence We may consider that the set of these p sequences is generated by the permutations

$$P_k = \begin{pmatrix} \theta_1(j) \\ \theta_k(j) \end{pmatrix}, \quad P_m = \begin{pmatrix} \theta_1(j) \\ \theta_m(j) \end{pmatrix} = \begin{pmatrix} \theta_k(j) \\ \theta_{(m,k)}(j) \end{pmatrix}$$

$$\therefore \quad P_m P_k = \begin{pmatrix} \theta_1(j) \\ \theta_{(m,k)}(j) \end{pmatrix} = P_{(m,k)} \tag{2.8}$$

Hence the permutations $P_1, \ldots, P_p$ form group P which is isomorphic to the group (2.4). For $l = 1, \ldots, p$, there are two such distinct permutation groups by virtue of (2). Apply to these groups the following property of the sequences (2.2):

(4) If l is odd, there is no pair of values of (k, j), $k > 1$, for which ${}^l\theta_k(j) = j$. If l is even, there exist pair of values of (k, j), $k > 1$, for which ${}^l\theta_k(j) = j$; for such values k is obviously even.

It follows from (4) that when l is odd, P is a transitive group and therefore a *regular permutation group* And when l is odd, P is intransitive. Owing to the isomorphism stated above, we may state that (Zassenhaus, 1949).

(5) P is a transitive representation of (2 4) when l is add, and an intransitive representation when l is even

3. We now introduce cyclic sequences of abstract elements, each with p distinct terms, by linking them up with the terms of the sequence (2 2). Denoting the elements by the b's, these sequences are

$$b^1_{\theta_1(j)} = b^1_j: \qquad b^1_1,\ b^1_2,\ \ldots,\ b^1_p \tag{3.1}$$

and the $p(p-1)$ sequences

$$ {}^l b^k_{{}^l\theta_k(j)}: \qquad {}^l b^k_{{}^l\theta_k(1)},\ {}^l b^k_{{}^l\theta_k(2)},\ \ldots,\ {}^l b^k_{{}^l\theta_k(p)}, \qquad \left.\begin{matrix} l = 1, \ldots, p \\ k = 2, \ldots, p \end{matrix}\right\} \pmod{p} \tag{3.2}$$

where, for each of these sequences, l and k remain fixed.

Consider an ordered set of p sequences given by (3.1) and the $p-1$ sequences given by $k = 2, \ldots, p$ and by a fixed velue of l arbitrarily chosen For the sake of simplicity, let this set be written as

$$ {}^t b^k_{\theta_k(j)}: \qquad {}^t b^k_{\theta_k(1)},\ {}^t b^k_{\theta_k(2)},\ \ldots,\ {}^t b^k_{\theta_k(p)}, \tag{3.3}$$

where the first sequence is given by $k = 1$ and by omitting altogether the left-hand index t; and the other sequences are given successively by $k = 2, \ldots, p$ and by $t = l$. Further, the right-hand lower symbols $\theta_k(j)$ are to be evaluated from (2.2) for the particular value of l chosen. Obviously, the ordered set (3.3) is a cyclic sequence of sequences and there are p such sets for $l = 1, \ldots, p$

Now let the p^2 terms (elements) of the p sequences (3.3) generate a set A in the following manner:

Let there exist a commutative composition, denoted by the symbol '$\circ$', whereby two elements of A are composed to form an element of A If $c, d, e, f, \ldots$ stand for the elements of A, the composition shall be governed by the following properties:

$$\left.\begin{array}{c} c \circ c = c, \text{ idempotent property} \\ \text{If } c \circ d = c \circ e, \text{ then } d = e \text{ cancellation property} \\ c \circ x = d \text{ has a unique solution in } A \end{array}\right\} \tag{3.4}$$

$$(c \circ d) \circ (e \circ f) = (c \circ e) \circ (d \circ f), \text{ entropic property} \tag{3.5}$$

With respect to this composition, the terms of the sequences (3.3) shall be governed by the following two properties: The element

$${}^{l}b_j^k \circ {}^{l}b_j^{k+p/2} = a_j \text{ remains invariant for all } k\ ; \tag{3.6}$$

and, with reference to the elements a_j thus obtained,

$${}^{l}b_j^k \circ b^1_{\theta_k(j)+p/2} = a_j \circ a_{\theta_k(j)+p/2}, \tag{3.7}$$

where the $\theta_k(j)$'s are to be evaluated from (2.2) for the particular value of l chosen.

Having thus introduced the sequences of elements, we notice in the first place that the above properties may be used to generate the sequences (3.3) as follows: Let us choose the kth and the $(k+p/2)$th sequences, say the 1st and the $(1+p/2)$th, arbitrarily. Given the composition, the sequence a_j can then be obtained from (3.6). It follows that the remaining sequences can then be obtained from (3.7). Or we may choose the cyclic sequence a_j and one of the sequences (3.3), say the 1st, arbitrarily. Then, given the composition, all the sequences (3.3) can be obtained from (3.7). It may be observed that for $k = 1+p/2$, the equations (3.6) and (3.7) become identical by virtue of (3.4).

Secondly, let us note how far the sequences (3.1) and (3.2) have been linked up with the sequences (2.2): Corresponding to the properties (1) and (2) of (2.2) given in §2, we can deduce the following properties for the sequences of elements (3.3):

(1) Of the $p(p-1)$ sequences (3.2), only $p+p/2-1$ are distinct for the following reasons: Given j arbitrarily, it follows from (3.7) that

(i) The elements ${}^{l}b_j^{2r+1}$ are equal to one another for a fixed value of r whatever value l may have.

(ii) For a fixed value of s and for different values of r, the elements ${}^{2s-2r+1}b_j^{2r}$ are equal to one another and the elements ${}^{2s-2r+2}b_j^{2r}$ are equal to one another.

(2) The p sets of ordered sequences (3.3), given by $l = 1, \ldots, p$, can be divided into two classes, according as l is odd or even, having the following property: Two sets of the same class can be transformed into one another by a suitable permutation of the sequences of a set; but this is not true of two sets belonging to different classes.

Having thus obtained the above properties corresponding to those in §2, it is but natural to inquire whether the sequences of an arbitrary set (3.3) form a group corresponding to the group (2.4) formed by the terms of the sequences (2.2). For this purpose it is necessary to obtain a formula connecting the terms of the sequences themselves. This can be done by eliminating the a's from the equations (3.7) with the help of the given composition in the following manner: It follows from (3.7), namely,

$${}^{l}b^k \circ b^1_{\theta_k(j)+p/2} = a_j \circ a_{\theta_k(j)+p/2}$$

that $\quad {}^t b^m_{\theta_k(j)} \circ b^1_{\theta_{(m,k)}(j)+p/2} = a_{\theta_k(j)} \circ a_{\theta_{(m,k)}(j)+p/2}$

therefore $\quad ({}^t b^k_j \circ b^1_{\theta_k(j)+p/2}) \circ ({}^t b^m_{\theta_k(j)} \circ b^1_{\theta_{(m,k)}(j)+p/2})$

$$= (a_j \circ a_{\theta_k(j)+p/2}) \circ (a_{\theta_k(j)} \circ a_{\theta_{(m,k)}(j)+p/2})$$

Using (3.5) and again (3.7), we hate

$$({}^t b^k_j \circ {}^t b^m_{\theta_k(j)}) \circ (b^1_{\theta_k(j)+p/2} \circ b^1_{\theta_{(m,k)}(j)+p/2})$$
$$= (a_j \circ a_{\theta_{(m,k)}(j)+p/2}) \circ (a_{\theta_k(j)} \circ a_{\theta_k(j)+p/2})$$
$$= ({}^t b_j^{(m,k)} \circ b^1_{\theta_{(m,k)}(j)+p/2}) \circ (b^1_{\theta_k(j)} \circ b^1_{\theta_k(j)+p/2})$$
$$= ({}^t b_j^{(m,k)} \circ b^1_{\theta_k(j)}) \circ (b^1_{\theta_k(j)+p/2} \circ b^1_{\theta_{(m,k)}(j)+p/2})$$

Therefore the required formula is

$${}^t b^k_j \circ {}^t b^m_{\theta_k(j)} = {}^t b_j^{(m,k)} \circ b^1_{\theta_k(j)}, \tag{3.8}$$

where it must be clearly understood that the left-hand upper index t of any term is to be altogether omitted if and only if the right-hand upper index of that term is equal to 1. Further, when $t = l$, the right-hand lower symbol $\theta_k(j)$ of that term is to be evaluated from (2.2) for the particular value of l chosen

The particular cases that follows from (3.8) are, by (2.5),

$${}^t b^k_j \circ {}^t b^k_{\theta_k(j)} = \begin{cases} {}^t b_j^{2k-1} \circ b^1_{\theta_k(j)}, & \text{if } k \text{ is odd} \\ b^1_j \circ b^1_{\theta_k(j)}, & \text{if } k \text{ is even} \end{cases} \tag{3.9}$$

and

$${}^l b_j^{1+p/2} \circ {}^l b^{1+p/2}_{j+p/2} = b^1_j \circ b^1_{j+p/2} \tag{3.10}$$

Now suppose the first sequence, denoted shortly by b^1, is a given one. It then follows from (3.4), (3.8), (3.9) and (3.10) that the remaining $p-1$ sequences of the set (3.3) can be determined. In general, the sequence b^1 having been already known, the equation (3.8) states that if the sequences ${}^t b^k$ and ${}^t b^m$ are given, the sequence ${}^t b^{(m,k)}$ becomes known by virtue of the property (3.4) of the composition provided that the terms of the sequences are taken as indicated by the lower indices of (3.8). It therefore follows that there exists a composition, denoted by dot ($\cdot$), say, which can be deduced from (3.8) and defined in such a manner that by means of this dot composition the sequences ${}^t b^k$ and ${}^t b^m$ can be composed to form the sequence ${}^t b^{(m,k)}$. Since it is seen from (3.8) that the particular terms of ${}^t b^m$ which are to be taken in this composition depend on ${}^t b^k$, it must be possible to write the dot composition (in accordance with the usual convention) as

$${}^t b^m . {}^t b^k = {}^t b^{(m,k)} \tag{3.11}$$

It must however be clearly understood that the equations (3.8) and (3.11) represent the same property. It follows evidently from (3.8) that the composition (3.11) is non-commutative.

Consider the following two equations that follow from (3.8):

$${}^t b^n_j \circ {}^t b^{(m,k)}_{\theta_n(j)} = {}^t b_j^{(n,(m,k))} \circ b^1_{\theta_n(j)}, \qquad {}^t b_j^{(n,m)} \circ {}^t b^k_{\theta_{(n,m)}(j)} = {}^t b_j^{((n,m),k)} \circ b^1_{\theta_{(n,m)}(j)}$$

As (3.8) is represented by (3.11), the above two equations must be represented respectively by

$${}^{t}b^{n}.({}^{t}b^{m}.{}^{t}b^{k}) = {}^{t}b^{(n,(m,k))}, \quad ({}^{t}b^{n}.{}^{t}b^{m}).{}^{t}b^{k} = {}^{t}b^{((n,m),k)}$$

It therefore follows from (2.7) that the composition (3.11) is associative. Further, it would follow from (2.5) that the inverse of ${}^{t}b^{k}$ is ${}^{t}b^{2-k}$ if k is odd and is ${}^{t}b^{k}$ if k is even. Hence, *the sequences of an arbitrary set* (3.3) *form a group* G, the unit element of this group being b^1 (the first sequence).

It also follows from (3.11) that this group is isomorphic to the group (2.4). Owing to this isomorphism, the property (3) of §2 hold here by virtue of the property (2) of this article, namely,

(3) Of the p groups of sequences (3.3) given by $l = 1, \ldots, p$, only two are distinct, one given by odd values of l and the other by even; and these two groups have $p/2$ sequences in common.

Having seen that the sequences of an arbitrary set (3.3) form a group G, it is again only natural to inquire whether there exists a group of transformations which shall be isomorphic to G corresponding to the group of permutations P of §2 which is isomorphic to the group (2.4).

Let there exist transformations, denoted by t_k, by which b^1 is transformed into b^k, $k = 1, \ldots, p$. As usual, let $t_m t_k$ denote the product of the transformations t_k and t_m when t_k is followed by t_m. Further, let us set the following property and the correspondence:

$$t_m t_k = t_{(m,k)} \quad \text{and} \quad t_k \longleftrightarrow b^k \tag{3.12}$$

Then the transformations t_k form a group T which is isomorphic to G. In T, let t_k^{-1} represent, as usual, the transformation transforming b^k into b^1. It then follows from (2.5) that

$$t_m t_k^{-1} = \begin{cases} t_m t_{2-k} = t_{(m,2-k)}, & \text{if } k \text{ is odd} \\ t_m t_k = t_{(m,k)}, & \text{if } k \text{ is even} \end{cases} \tag{3.13}$$

represents the transformation by which b^k is transformed into b^m. We shall find it convenient to put

$$t_{k+1} t_k^{-1} = T_{k+1,k} \tag{3.14}$$

and shall discuss the properties, under certain circumstances, of the transformations $T_{k+1,k}$ by which b^k is transformed into b^{k+1} in the next article.

We give below certain other equations for the sequences (3.3) derived from (3.6) and (3.7) which we shall consider later in §5 It follows from (3.5) and (3.6) that

$$a_j \circ a_{j+p/2} = ({}^{t}b_j^{k} \circ {}^{l}b_j^{k+p/2}) \circ ({}^{t}b_{j+p/2}^{k} \circ {}^{l}b_{j+p/2}^{k+p/2}) = ({}^{t}b_j^{k} \circ {}^{t}b_{j+p/2}^{k}) \circ ({}^{l}b_j^{k+p/2} \circ {}^{l}b_{j+p/2}^{k+p/2}).$$

In the same manner

$$a_j \circ a_{\theta_k(j)+p/2} = ({}^{t}b_j^{k} \circ b_{\theta_k(j)+p/2}^{1}) \circ ({}^{l}b_j^{k+p/2} \circ {}^{l}b_{\theta_k(j)+p/2}^{1+p/2}).$$

Comparing with (3.7),

$$a_j \circ a_{\theta_k(j)+p/2} = {}^{t}b_j^{k} \circ b_{\theta_k(j)+p/2}^{1} = {}^{l}b_j^{k+p/2} \circ {}^{l}b_{\theta_k(j)+p/2}^{1+p/2}. \tag{3.15}$$

Putting $k = 1$,

$$a_j \circ a_{j+p/2} = b_j^1 \circ b_{j+p/2}^1 = {}^l b_j^{1+p/2} \circ {}^l b_{j+p/2}^{1+p/2}.$$

This gives (3.10) in terms of the a's. Thus, the two equations that come out are (3.15) and

$$a_j \circ a_{j+p/2} = b_j^1 \circ b_{j+p/2}^1 = {}^l b_j^{1+p/2} \circ {}^l b_{j+p/2}^{1+p/2} = ({}^l b_j^k \circ {}^l b_{j+p/2}^k) \circ ({}^l b_j^{k+p/2} \circ {}^l b_{j+p/2}^{k+p/2}),$$
$$k = 2, \ldots, p. \tag{3.16}$$

4. We shall now introduce a special type of properties by means of which all the p sets of ordered sequences of elements (3.3) considered in the last article may be constructed, as a special type, satisfying all the properties mentioned there. For this purpose, we first link up the sequence (3.1) with the sequence (2.2) as follows:

Let b be an abstract element and suppose that there exist two operations, denoted by $*$, $'$, which can be applied to b so as to obtain two other elements, denoted by b^* and b'. Further suppose that these operations can be applied in the same way to b^* and b' and to every element generated in this manner so as to obtain a set of elements. Moreover, let the two operations satisfy the properties (3) and (4) of §1, namely that if x is an element of the set, then

$$x^{**} = x'' = x \tag{4.1}$$

and that the sequence of elements

$$b_1 = b, \quad b_2 = b^*, \quad b_3 = b^{*\prime}, \quad b_4 = b^{*\prime *}, \ldots \tag{4.2}$$

is a finite, and therefore a cyclic, sequence with p distinct terms, where $p > 4$ is the integer of the type $p = 4n$ as chosen in §1(†).

Furthermore, suppose that there exists a commutative composition whereby two elements b_i and b_j of (4.2) can be composed to form an element $b_i \circ b_j$ to which also repeated operations of $*$ and $'$ can be applied so as to obtain other elements, and that the composition has the property

$$(b_i \circ b_j)^* = b_i^* \circ b_j^*, \quad (b_i \circ b_j)' = b_i' \circ b_j'. \tag{4.3}$$

We now identify (3.1), *i.e.* the first sequence of (3.3), with the sequence (4.2) by setting

$$b_1 = b_1^1, \; b_2 = b_2^1, \; b_3 = b_3^1, \; \ldots, \; b_p = b_p^1$$

and also identify the composition given above with that defined by (3.4) and (3.5).

As in the last article, consider a particular ordered set (3.3) of p sequences given by a particular value of l chosen arbitrarily and let the successive sequences of this set be denoted by $\Sigma_1, \Sigma_2, \ldots, \Sigma_p$ which form, as we know, a cyclic sequence. It follows from (3.14) that Σ_1 is transformed into Σ_2 by the transformation $T_{2,1}$, Σ_2 into Σ_3 by $T_{3,2}, \ldots$, Σ_{p-1} into Σ_p by $T_{p,p-1}$ and Σ_p into Σ_1 by $T_{1,p}$. From the nature of the sequences themselves we shall assume certain properties of these transformations and deduce from them certain consequence.

We may regard the transformation $T_{2,1}$ as transforming b_j^1 into ${}^l b_{\theta_2(j)}^2$. We may therefore assume that this transformation has the property that it transforms the (first) term b_1^1 of Σ_1 into the (first) term ${}^l b_{\theta_2(1)}^2$ of Σ_2 and at the same time transforms the pair of

(†) It may be mentioned here that the terms of the sequence (4.2) would form a group corresponding to the group (2.4) provided that the element b can be regarded as a symbol for the identity operation.

operations $(*, ')$ into a pair of operations, say $(^+, ^-)$, and leaves the composition invariant. Then Σ_2 is considered to be generated by $(^+, ^-)$ in the same manner as Σ_1 is generated by $(*, ')$. The operations $^+, ^-$ however are not always different from $*, '$, as will appear below. Thus, by application of $T_{2,1}$, from

$$b_j^{1**} = b_j^{1''} = b_j^1 \text{ and } (b_i^1 \circ b_j^1)^* = b_i^{1*} \circ b_j^{1*}, \ (b_i^1 \circ b_j^1)' = b_i^{1'} \circ b_j^{1'}$$

we obtain

$$b_j^{1*} \to {}^l b^{2+}_{\theta_2(j)}, \ b_j^{1'} \to {}^l b^{2-}_{\theta_2(j)}, \ {}^l b^{2++}_{\theta_2(j)} = {}^l b^{2--}_{\theta_2(j)} = {}^l b^{2}_{\theta_2(j)}$$

and

$$({}^l b^2_{\theta_2(i)} \circ {}^l b^2_{\theta_2(j)})^+ = {}^l b^{2+}_{\theta_2(i)} \circ {}^l b^{2+}_{\theta_2(j)}, \ ({}^l b^2_{\theta_2(i)} \circ {}^l b^2_{\theta_2(j)})^- = {}^l b^{2-}_{\theta_2(i)} \circ {}^l b^{2-}_{\theta_2(j)}.$$

We shall assume that the above property holds for every transformation under consideration. Thus, for $k = 1, 2, \ldots$

(I) The transformation $T_{k+1,k}$ transforms a term of Σ_k into the corresponding term of Σ_{k+1} and at the same time transforms the ordered pair of operations by which Σ_k is generated into an ordered pair of operations by which Σ_{k+1} is generated and leaves the composition invariant.

It follows that if Σ_m is generated by $(^+, ^-)$, then this pair of operations has the properties (4.1) and (4.3), namely

(1) $${}^l b_j^{m++} = {}^l b_j^{m--} = {}^l b_j^m, \ ({}^l b_i^m \circ {}^l b_j^m)^+ = {}^l b_i^{m+} \circ {}^l b_j^{m+}, \ ({}^l b_i^m \circ {}^l b_j^m)^- = {}^l b_i^{m-} \circ {}^l b_j^{m-}.$$

Now each of the cyclic sequences $\Sigma_1, \ldots, \Sigma_p$ has the one or the other of two opposite senses corresponding to the two cyclic senses (orders) of the lower indices of its successive terms. From the construction (4.2) of Σ_1 it is seen that the operation $*$ has been chosen with the property that when it is applied to a term of Σ_1 having an odd (even) lower index, it transforms the term into a term of Σ_1 having the next greater (lesser) lower index (mod. p). Analogous consideration holds for the operation $'$. This leads us to assume the following invariant character of a pair of operations under the transformations $T_{k+1,k}$, $k = 1, 2, \ldots$, on the supposition of getting not more than two pairs of operations.

(II) The sequences Σ_k and Σ_{k+1} have the same (opposite) sense when $\theta_k(j)$ and $\theta_{k+1}(j)$ are both odd or both even (one odd, one even) if and only if the pair of operations by which Σ_k is generated remains invariant under the transformation $T_{k+1,k}$.

There remains therefore just one alternative for which the pair of operations does not remain invariant for $T_{k+1,k}$. Accordingly, the transformations have the property that a sequence Σ_k is generated either by the pair of operations $(*, ')$ (by which Σ_1 is generated) or by a *different* pair of operations, say $(^+, ^-)$.

We are now in a position to determine which one of the two different pairs of operations generates a particular sequence. This can be done by comparing the sense of the particular sequence with that of Σ_1. Referring to the sequence (2.2) the following cases arise:

(a) k is odd

In this case j and $\theta_k(j)$ are both odd or both even, and Σ_1 and Σ_k have the same sense. Therefore Σ_k is generated by $(^*, ')$

(b) k is even

(i) l is odd. Here one of j and $\theta_k(j)$ is odd and the other even, and Σ_1 and Σ_k have opposite senses. Therefore Σ_k is generated by $(^*, ')$.

(ii) l is even. Here j and $\theta_k(j)$ are both odd or both even, and Σ_1 and Σ_k have opposite senses Therefore Σ_k is generated by $(^+, ^-)$

We thus derive the following conclusion:

(2) When l is odd, all the p sequences of the set (3.8) are generated by $(^*, ')$; and when l is even, $p/2$ of the sequences are generated by $(^*, ')$ and the remaining $p/2$ by $(^+, ^-)$. Since, by property (3) of §3, these two classes of sequences have $p/2$ sequences in common, the common sequences must necessarily be the sequences Σ_k, where k is odd.

As stated at the begining of this article, it is our intention to construct the sequences introduced in the last article having all the properties mentioned there, the central property being the equation (3.7). Remembering that the lower indices of the equations (3.7), (3.8), (3.9) are given by the terms of the sequence (2.2) and that the construction of (2.2) is given by (1.1) and (1.2), it is seen that (3.9) and (3.8) give respectively

$$b_j^1 \circ b_{\psi(j)}^1 = {}^lb_j^2 \circ {}^lb_{\psi(j)}^2$$

and

$${}^lb_j^p \circ {}^lb_{\chi(j)}^2 = {}^lb_j^3 \circ b_{\chi(j)}^1$$

writing $\chi(j)$ for j,

$${}^lb_{\chi(j)}^p \circ {}^lb_j^2 = {}^lb_{\chi(j)}^3 \circ b_j^1$$

Using (3.5),

$$({}^lb_j^2 \circ {}^lb_{\chi(j)}^2) \circ ({}^lb_j^p \circ {}^lb_{\chi(j)}^p) = ({}^lb_j^3 \circ {}^lb_{\chi(j)}^3) \circ (b_j^1 \circ b_{\chi(j)}^1)$$

But (3.8) gives also

$${}^lb_j^p \circ {}^lb_{\chi(j)}^p = b_j^1 \circ b_{\chi(j)}^1$$

Therefore

$${}^lb_j^2 \circ {}^lb_{\chi(j)}^2 = {}^lb_j^3 \circ {}^lb_{\chi(j)}^3$$

In a similar manner

$${}^lb_j^3 \circ {}^lb_{\psi(j)}^3 = {}^lb_j^1 \circ {}^lb_{\psi(j)}^4.$$

And so on.

We may therefore make the following assumptions for the transformation $T_{k+1,k}$, $k = 1, 2, \ldots$,

(III) The transformation $T_{k+1,k}$ has the property that

$${}^lb_j^k \circ {}^lb_{\psi(j)}^k = {}^lb_j^{k+1} \circ {}^lb_{\psi(j)}^{k+1} \quad \text{or} \quad {}^lb_j^k \circ {}^lb_{\chi(j)}^k = {}^lb_j^{k+1} \circ {}^lb_{\chi(j)}^{k+1}, \tag{4.4}$$

according as k is odd or even. It therefore follows that

(3) (i) if l and k are both odd,

$$({}^lb_j^k \circ {}^lb_{\psi(j)}^k)^* = ({}^lb_j^{k+1} \circ {}^lb_{\psi(j)}^{k+1})^*, \quad ({}^lb_j^k \circ {}^lb_{\psi(j)}^k)' = ({}^lb_j^{k+1} \circ {}^lb_{\psi(j)}^{k+1})'$$

(*ii*) if l is odd and k is even,

$$({}^lb_j^k \circ {}^lb^k_{\chi(j)})^* = ({}^lb_j^{k+1} \circ {}^lb^{k+1}_{\chi(j)})^*, \quad ({}^lb_j^k \circ {}^lb^k_{\chi(j)})' = ({}^lb_j^{k+1} \circ {}^lb^{k+1}_{\chi(j)})'$$

(*iii*) if l is even and k is odd,

$$({}^lb_j^k \circ {}^lb^k_{\psi(j)})^* = ({}^lb_j^{k+1} \circ {}^lb^{k+1}_{\psi(j)})^+, \quad ({}^lb_j^k \circ {}^lb^k_{\psi(j)})' = ({}^lb_j^{k+1} \circ {}^lb^{k+1}_{\psi(j)})^-$$

(*iv*) if l and k are both even,

$$({}^lb_j^k \circ {}^lb^k_{\chi(j)})^+ = ({}^lb_j^{k+1} \circ {}^lb^{k+1}_{\chi(j)})^*, \quad ({}^lb_j^k \circ {}^lb^k_{\chi(j)})^- = ({}^lb_j^{k+1} \circ {}^lb^{k+1}_{\chi(j)})'$$

The consequences (1), (2) and (3) that follow from the assumptions (I), (II) and (III) regarding the properties of the transformations $T_{k+1,k}$, together with the equation (3.6), will enable us to construct the special type of the p sets of sequences (3.3) satisfying the equation (3.7). Before the actual construction is undertaken, we take note in the next article of a special property which these special sequences may possess.

5. Consider the two pairs of operations and the composition as defined by their properties in the last article. Take any sequence Σ_m of (3.3) which is supposed to be generated by $*$, $'$ (or $^+$, $^-$) and a particular term (element), say the first, of Σ_m. Let this element generate a system of elements defined by the following properties:

(*i*) If ξ is an element of the system, then ξ^*, ξ'(or ξ^+, ξ^-) are elements of the system.

(*ii*) If ξ and η are two elements of the system, then $\xi \circ \eta$ is an element of the system.

(*iii*) $\xi \circ \xi = \xi$. If $\xi \circ \eta = \xi \circ \zeta$, then $\eta = \zeta$.

It is then known (Sen, 1950) that under certain circumstances this system has a unique element, say u, possessing the following property:

$$u^* = u' = u \text{ (or } u^+ = u^- = u) \tag{5.1}$$

Now let the ordered sequences (3.3) for a particular odd value of l be denoted by $\Sigma_1, \Sigma_2, \ldots, \Sigma_p$ and those for a particular even value of l be denoted by $\Sigma_1, \bar{\Sigma}_2, \Sigma_3, \bar{\Sigma}_4, \Sigma_5, \ldots, \bar{\Sigma}_p$. The odd sequences are not altered by virtue of (1) (*i*) of §3. Then, by (2) of the last article, the Σ's without bar are generated by ($*$, $'$) while those with bar are generated by ($^+$, $^-$). Further, let the systems generated in the manner stated above by particular elements, say the first, of these sequences be denoted by $S_1, S_2, \ldots, S_p$ and $S_1, \bar{S}_2, S_3, \bar{S}_4, \ldots, \bar{S}_p$ respectively. Moreover, by what we have stated above, let the unique elements of these systems be denoted by $u^1, u^2, \ldots u^p$ and $u^1, \bar{u}^2, u^3, \bar{u}^4, \ldots, \bar{u}^p$ respectively. It is then known that under certain circumstances these unique elements are given by (3.16). That is,

$$u^1 = u^{1+p/2} = u^2 \circ u^{2+p/2} = u^3 \circ u^{3+p/2} = \ldots = \bar{u}^2 \circ \bar{u}^{2+p/2} = \bar{u}^4 \circ \bar{u}^{4+p/2} = \ldots \tag{5.2}$$

The special type of sequences which are given in article §7 has the special property that

the unique elements of all the derived systems $S_1, S_2, \ldots, S_p, \bar{S}_2, \bar{S}_4, \ldots, \bar{S}_p$ are equal to one another. That is

$$u = u^1 = u^2 = \cdots = u^p = \bar{u}^2 = \bar{u}^4 = \cdots = \bar{u}^p \tag{5.3}$$

The reasons for the existence of this common property requires investigation which, however, shall not be attempted here. We shall merely state that

(1) The join or the union of the derived systems may be regarded as one algebraic system of elements generated by a particular term, say the first, of Σ_1 and constructed with the help of the pair of operations (*, $'$), the composition '$\circ$' and the transformations $T_{k+1,k}$, and that the system has the unique element u satisfying both the parts of (5.1).

Finally, we may mention the following interesting point of view. We have seen in §3 that the sequences $\Sigma_1, \Sigma_2, \ldots, \Sigma_p$ form a group, so do the sequences $\Sigma_1, \bar{\Sigma}_2, \Sigma_3, \ldots, \bar{\Sigma}_p$ and these sequences generate the derived systems $S_1, S_2, \ldots, S_p$ and $S_1, \bar{S}_2, S_3, \ldots \bar{S}_p$. There is therefore no harm to say that

(2) The systems $S_1, S_2, \ldots, S_p$ form a group and therefore an associative system although the systems themselves are not necessarily associative. The same thing holds for $S_1, \bar{S}_2, S_3, \ldots \bar{S}_p$.

6. All the properties of the sequences and the systems of elements that we have discussed so far have interesting applications in Piemannian geemetry. In giving the applications, the usual notations of tensor calculus are adopted.

Let the metric of a Riemannian space be given, as usual, by

$$ds^2 = g_{ij}dx^i dx^j$$

and in this space let there exist a law of parallel displacement of contravariant vectors defined, as usual, by

$$dV^t + \Gamma^t_{ij} V^i dx^j = 0, \tag{6.1}$$

where the coefficients of affine connection Γ^t_{ij} are supposed to be arbitrary. Further, let the covariant derivatives of tensors with respect to the parallel displacement (6.1) (corresponding to Γ^t_{ij}) be denoted by a comma followed by indices. Put

$$a = \Gamma^t_{ij}, \quad a^* = \Gamma^t_{ij} + g^{mt} g_{im,j}, \quad a' = \Gamma^t_{ji}. \tag{6.2}$$

Obviously, a^* and a' are two sets of coefficients of affine connections. If the covariant derivatives of tensors with respect to (6.1) corresponding to a^* (in place of a) be denoted by a semi-colon followed by indices, it may be seen that

$$g_{im,j} = g_{im;j} = 0.$$

It therefore follows that

$$a^{**} = a'' = a. \tag{6.3}$$

Now construct the following sequence, say Σ,

$$a_j: \qquad a_1 = a, \quad a_2 = a^*, \quad a_3 = a^{*\prime}, \quad a_4 = a^{*\prime *}, \ldots \tag{6.4}$$

It is then known (Sen, 1950) that Σ is a cyclic sequence with $p = 12$ terms which are

supposed to be distinct. It is further known that if $c = L^t_{ij}$ is another set of coefficients of affine connections, then

$$[\tfrac{1}{2}(a+c)]^* = \tfrac{1}{2}(a^*+c^*), \quad [\tfrac{1}{2}(a+c)]' = \tfrac{1}{2}(a'+c') \tag{6.5}$$

As a matter of fact, the second equation is obvious and the first can be established by forming covariant derivatives of the g_{ij}'s with respect to (6.1) corresponding to a, c and $\frac{1}{2}(a+c)$.

Thus, the pair of operations (*, ′) has been defined by (6.2) satisfying (6.3), *i.e.*, (4.1), and the composition is defined as half the sum satisfying (3.4), (3.5) and (6.5), *i.e.* (4.3).

As in the last article, form the system S generated by a having the following properties :

(*i*) If ξ is an element of S, then ξ^* and ξ' are elements of S.

(*ii*) If ξ and η are elements of S, then $\frac{1}{2}(\xi+\eta)$ is an element of S. It is then known that the christoffel symbol is given by

$$\left.\begin{aligned} &\begin{Bmatrix} t \\ ij \end{Bmatrix} = \tfrac{1}{2}(a_j + a_{j+6}), \quad j = 1, 2, \ldots \\ &\text{and it is the unique element of } S \text{ having the property} \\ &\begin{Bmatrix} t \\ ij \end{Bmatrix}^* = \begin{Bmatrix} t \\ ij \end{Bmatrix}' = \begin{Bmatrix} t \\ ij \end{Bmatrix} \end{aligned}\right\} \tag{6.6}$$

7. We shall now obtain the 12 sets of 12 ordered sequences (3.3). It is only necessary to find two such sets for two particular values of l, one odd and the other even, which, without loss of generality, we take to be $l = 1$ and $l = 2$. The other sets can then be easily deduced.

Let the ordered sequences (3.3) for the value $l = 1$ be denoted by $\Sigma_1, \ldots, \Sigma_{12}$. The sequence (2.2) for $l = 1$ is

$$\left.\begin{aligned} &\theta_1(j) = j, \ \theta_2(j) = 1-j, \ \theta_3(j) = 2+j, \ \theta_4(j) = 3-j, \ \theta_5(j) = 4+j, \ \theta_6(j) = 5-j \\ &\theta_7(j) = 6+j, \ \theta_8(j) = 7-j, \ \theta_9(j) = 8+j, \ \theta_{10}(j) = 9-j, \ \theta_{11}(j) = 10+j, \ \theta_{12}(j) = 11-j \end{aligned}\right\} \tag{7.1}$$

Let T^t_{ij} be an arbitrary tensor. Put

$$\gamma = T^t_{ij}, \ \delta = g^{mt}g_{is}T^s_{mj}, \ \epsilon = g^{mt}g_{is}T^s_{jm}, \ \gamma_c = T^t_{ji}, \ \delta_c = g^{mt}g_{js}T^s_{mi}, \ \epsilon_c = g^{mt}g_{js}T^s_{im}. \tag{7.2}$$

Take 12 sets of coefficients of affine connections defined as follows :

$$b^1 = a+\gamma, \ {}^1b^2 = a+\gamma_c, \ {}^1b^3 = a-\delta_c, \ {}^1b^4 = a-\delta, \ {}^1b^5 = a+\epsilon, \ {}^1b^6 = a+\epsilon_c$$

$${}^1b^7 = a-\gamma, \ {}^1b^8 = a-\gamma_c, \ {}^1b^9 = a+\delta_c, \ {}^1b^{10} = a+\delta, \ {}^1b^{11} = a-\epsilon, \ {}^1b^{12} = a-\epsilon_c$$

and construct the sequences generated by

$$b^1 = b^1_1, \ {}^1b^2 = {}^1b^2_1, \ {}^1b^3 = {}^1b^3_1, \ldots, \ {}^1b^{12} = {}^1b^{12}_1$$

with the help of the pair of operations (*, ′) in the same manner as the sequence (6.4) is generated by a. By straightforward calculation we find the following values of the terms of the 12 sequences written vertically :

$b^1_1 = a_1+\gamma$	$^1b^2_1 = a_1+\gamma_c$	$^1b^3_1 = a_1-\delta_c$	$^1b^4_1 = a_1-\delta$	$^1b^5_1 = a_1+\epsilon$	$^1b^6_1 = a_1+\epsilon_c$
$b^1_2 = a_2-\delta$	$^1b^2_2 = a_2-\epsilon$	$^1b^3_2 = a_2+\epsilon_c$	$^1b^4_2 = a_2+\gamma$	$^1b^5_2 = a_2-\gamma_c$	$^1b^6_2 = a_2-\delta_c$
$b^1_3 = a_3-\delta_c$	$^1b^2_3 = a_3-\epsilon_c$	$^1b^3_3 = a_3+\epsilon$	$^1b^4_3 = a_3+\gamma_c$	$^1b^5_3 = a_3-\gamma$	$^1b^6_3 = a_3-\delta$
$b^1_4 = a_4+\epsilon_c$	$^1b^2_4 = a_4+\delta_c$	$^1b^3_4 = a_4-\gamma_c$	$^1b^4_4 = a_4-\epsilon$	$^1b^5_4 = a_4+\delta$	$^1b^6_4 = a_4+\gamma$
$b^1_5 = a_5+\epsilon$	$^1b^2_5 = a_5+\delta$	$^1b^3_5 = a_5-\gamma$	$^1b^4_5 = a_5-\epsilon_c$	$^1b^5_5 = a_5+\delta_c$	$^1b^6_5 = a_5+\gamma_c$
$b^1_6 = a_6-\gamma_c$	$^1b^2_6 = a_6-\gamma$	$^1b^3_6 = a_6+\delta$	$^1b^4_6 = a_6+\delta_c$	$^1b^5_6 = a_6-\epsilon_c$	$^1b^6_6 = a_6-\epsilon$
$b^1_7 = a_7-\gamma$	$^1b^2_7 = a_7-\gamma_c$	$^1b^3_7 = a_7+\delta_c$	$^1b^4_7 = a_7+\delta$	$^1b^5_7 = a_7-\epsilon$	$^1b^6_7 = a_7-\epsilon_c$
$b^1_8 = a_8+\delta$	$^1b^2_8 = a_8+\epsilon$	$^1b^3_8 = a_8-\epsilon_c$	$^1b^4_8 = a_8-\gamma$	$^1b^5_8 = a_8+\gamma_c$	$^1b^6_8 = a_8+\delta_c$
$b^1_9 = a_9+\delta_c$	$^1b^2_9 = a_9+\epsilon_c$	$^1b^3_9 = a_9-\epsilon$	$^1b^4_9 = a_9-\gamma_c$	$^1b^5_9 = a_9+\gamma$	$^1b^6_9 = a_9+\delta$
$b^1_{10} = a_{10}-\epsilon_c$	$^1b^2_{10} = a_{10}-\delta_c$	$^1b^3_{10} = a_{10}+\gamma_c$	$^1b^4_{10} = a_{10}+\epsilon$	$^1b^5_{10} = a_{10}-\delta$	$^1b^6_{10} = a_{10}-\gamma$
$b^1_{11} = a_{11}-\epsilon$	$^1b^2_{11} = a_{11}-\delta$	$^1b^3_{11} = a_{11}+\gamma$	$^1b^4_{11} = a_{11}+\epsilon_c$	$^1b^5_{11} = a_{11}-\delta_c$	$^1b^6_{11} = a_{11}-\gamma_c$
$b^1_{12} = a_{12}+\gamma_c$	$^1b^2_{12} = a_{12}+\gamma$	$^1b^3_{12} = a_{12}-\delta$	$^1b^4_{12} = a_{12}-\delta_c$	$^1b^5_{12} = a_{12}+\epsilon_c$	$^1b^6_{12} = a_{12}+\epsilon$

(7.3)

$^1b^7_1 = a_1-\gamma$	$^1b^8_1 = a_1-\gamma_c$	$^1b^9_1 = a_1+\delta_c$	$^1b^{10}_1 = a_1+\delta$	$^1b^{11}_1 = a_1-\epsilon$	$^1b^{12}_1 = a_1-\epsilon_c$
$^1b^7_2 = a_2-\delta$	$^1b^8_2 = a_2+\epsilon$	$^1b^9_2 = a_2-\epsilon_c$	$^1b^{10}_2 = a_2-\gamma$	$^1b^{11}_2 = a_2+\gamma_c$	$^1b^{12}_2 = a_2+\delta_c$
$^1b^7_3 = a_3+\delta_c$	$^1b^8_3 = a_3+\epsilon_c$	$^1b^9_3 = a_3-\epsilon$	$^1b^{10}_3 = a_3-\gamma_c$	$^1b^{11}_3 = a_3+\gamma$	$^1b^{12}_3 = a_3+\delta$
$^1b^7_4 = a_4-\epsilon_c$	$^1b^8_4 = a_4-\delta_c$	$^1b^9_4 = a_4+\gamma_c$	$^1b^{10}_4 = a_4+\epsilon$	$^1b^{11}_4 = a_4-\delta$	$^1b^{12}_4 = a_4-\gamma$
$^1b^7_5 = a_5-\epsilon$	$^1b^8_5 = a_5-\delta$	$^1b^9_5 = a_5+\gamma$	$^1b^{10}_5 = a_5+\epsilon_c$	$^1b^{11}_5 = a_5-\delta_c$	$^1b^{12}_5 = a_5-\gamma_c$
$^1b^7_6 = a_6+\gamma_c$	$^1b^8_6 = a_6+\gamma$	$^1b^9_6 = a_6-\delta$	$^1b^{10}_6 = a_6-\delta_c$	$^1b^{11}_6 = a_6+\epsilon_c$	$^1b^{12}_6 = a_6+\epsilon$
$^1b^7_7 = a_7+\gamma$	$^1b^8_7 = a_7+\gamma_c$	$^1b^9_7 = a_7-\delta_c$	$^1b^{10}_7 = a_7-\delta$	$^1b^{11}_7 = a_7+\epsilon$	$^1b^{12}_7 = a_7+\epsilon_c$
$^1b^7_8 = a_8-\delta$	$^1b^8_8 = a_8-\epsilon$	$^1b^9_8 = a_8+\epsilon_c$	$^1b^{10}_8 = a_8+\gamma$	$^1b^{11}_8 = a_8-\gamma_c$	$^1b^{12}_8 = a_8-\delta_c$
$^1b^7_9 = a_9-\delta_c$	$^1b^8_9 = a_9-\epsilon_c$	$^1b^9_9 = a_9+\epsilon$	$^1b^{10}_9 = a_9+\gamma_c$	$^1b^{11}_9 = a_9-\gamma$	$^1b^{12}_9 = a_9-\delta$
$^1b^7_{10} = a_{10}+\epsilon_c$	$^1b^8_{10} = a_{10}+\delta_c$	$^1b^9_{10} = a_{10}-\gamma_c$	$^1b^{10}_{10} = a_{10}-\epsilon$	$^1b^{11}_{10} = a_{10}+\delta$	$^1b^{12}_{10} = a_{10}+\gamma$
$^1b^7_{11} = a_{11}+\epsilon$	$^1b^8_{11} = a_{11}+\delta$	$^1b^9_{11} = a_{11}-\gamma$	$^1b^{10}_{11} = a_{11}-\epsilon_c$	$^1b^{11}_{11} = a_{11}+\delta_c$	$^1b^{12}_{11} = a_{11}+\gamma_c$
$^1b^7_{12} = a_{12}-\gamma_c$	$^1b^8_{12} = a_{12}-\gamma$	$^1b^9_{12} = a_{12}+\delta$	$^1b^{10}_{12} = a_{12}+\delta_c$	$^1b^{11}_{12} = a_{12}-\epsilon_c$	$^1b^{12}_{12} = a_{12}-\epsilon$

The first of the above 12 sequences (7.3) is Σ_1; the terms of each of the other sequences have to be rearranged according to the order of the lower indices of the b's in (3.3)

and evaluated from (7.1) in order to obtain $\Sigma_2, \ldots, \Sigma_{12}$. It may now be verified that the equations (3.6) to (3.10) and (4.4) are all satisfied for (7.3) ; also 3(*i*), (*ii*) §4 hold.

It follows from (3.8), namely

$$\tfrac{1}{2}({}^t b^k_j + {}^t b^m_{\theta_k(j)}) = \tfrac{1}{2}({}^t b^{(m,k)}_j + b^1_{\theta_k(j)})$$

that

$${}^t b^k_j + {}^t b^m_{\theta_k(j)} - b^1_{\theta_k(j)} = {}^t b^{(m,k)}_j. \tag{7.4}$$

This equation (7.4) corresponds to the equation (3.11) of group-composition of a set of sequences. From (7.1) and (7.4), the multiplication table of the group of sequences (7.3) is seen to be

$$\begin{array}{c|ccccc} & \Sigma_1 & \Sigma_2 & \Sigma_3 & \ldots & \Sigma_{12} \\ \hline \Sigma_1 & \Sigma_1 & \Sigma_2 & \Sigma_3 & \ldots & \Sigma_{12} \\ \Sigma_2 & \Sigma_2 & \Sigma_1 & \Sigma_{12} & \ldots & \Sigma_3 \\ \Sigma_3 & \Sigma_3 & \Sigma_4 & \Sigma_5 & \ldots & \Sigma_2 \\ \cdot\,\cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \Sigma_{12} & \Sigma_{12} & \Sigma_{11} & \Sigma_{10} & \ldots & \Sigma_1 \end{array} \tag{7.5}$$

The group (7.5) is isomorphic to the group (2.4) for $p = 12$, as is to be expected. This group can be put in a $1-1$ correspondence with a permutation group as follows. Denote the tensors (7.2) as

$$\gamma_1 = \gamma,\ \gamma_2 = -\delta,\ \gamma_3 = -\delta_c,\ \gamma_4 = \epsilon_0,\ \gamma_5 = \epsilon,\ \gamma_6 = -\gamma_0$$

$$\gamma_7 = -\gamma,\ \gamma_8 = \delta,\ \gamma_9 = \delta_c,\ \gamma_{10} = -\epsilon_0,\ \gamma_{11} = -\epsilon,\ \gamma_{12} = \gamma_c$$

and consider the following 12 permutations out of the symmetric group of permutations of the above 12 tensors :

$$\begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{12} \\ \gamma_{i_1} & \gamma_{i_2} & \cdots & \gamma_{i_{12}} \end{pmatrix} \tag{7.6}$$

where the successive tensors in the first line of (7.6) are the successive tensors given in the first sequence of (7.3) and the successive tensors in the second line are similarly taken as those given in the 12 sequences (7.3). It may than be seen that the 12 permutations (7.6) form a group which is isomorphic to the group P of §2 for $p = 12$ and is therefore, by property (5) §2, a transitive representation of the group (7.5).

Let $S_1, S_2, \ldots, S_{12}$ be the systems of elements derived from the sequences $\Sigma_1, \Sigma_2, \ldots, \Sigma_{12}$ respectively in the manner described in §5. It may then be verified from (6.6) and (7.5) that the statements (1) and (2) of §5 hold, namely that the unique elements of these systems are equal to the christoffel symbol and that the systems, which are themselves non-associative, form a group which is obtained from (7.5) by replacing the Σ's by the S's.

Let us now consider the case $l = 2$. Let the ordered sequences (3.3) be denoted in this case by $\Sigma_1, \bar{\Sigma}_2, \Sigma_3, \bar{\Sigma}_4, \ldots, \bar{\Sigma}_{12}$. As the Σ's without bar have been obtained before,

we have only to get the 6 Σ's with bar. They are (except for rearrangement of the terms according to the lower indices of the b's in (3.3) for $l = 2$) as follows: (7.7)

${}^2b_1^2 = a_1+\gamma$	${}^2b_1^4 = a_1-\delta_c$	${}^2b_1^6 = a_1+\epsilon$	${}^2b_1^8 = a_1-\gamma$	${}^2b_1^{10} = a_1+\delta_c$	${}^2b_1^{12} = a_1-\epsilon$
${}^2b_2^2 = a_2+\gamma_c$	${}^2b_2^4 = a_2-\delta$	${}^2b_2^6 = a_2+\epsilon_c$	${}^2b_2^8 = a_2-\gamma_c$	${}^2b_2^{10} = a_2+\delta$	${}^2b_2^{12} = a_2-\epsilon_c$
${}^2b_3^2 = a_3-\epsilon$	${}^2b_3^4 = a_3+\gamma$	${}^2b_3^6 = a_3-\delta_c$	${}^2b_3^8 = a_3+\epsilon$	${}^2b_3^{10} = a_3-\gamma$	${}^2b_3^{12} = a_3+\delta_c$
${}^2b_4^2 = a_4-\epsilon_c$	${}^2b_4^4 = a_4+\gamma_c$	${}^2b_4^0 = a_4-\delta$	${}^2b_4^8 = a_4+\epsilon_c$	${}^2b_4^{10} = a_4-\gamma_c$	${}^2b_4^{12} = a_4+\delta$
${}^2b_5^2 = a_5+\delta_c$	${}^2b_5^4 = a_5-\epsilon$	${}^2b_5^6 = a_5+\gamma$	${}^2b_5^8 = a_5-\delta_c$	${}^2b_5^{10} = a_5+\epsilon$	${}^2b_5^{12} = a_5-\gamma$
${}^2b_6^2 = a_6+\delta$	${}^2b_6^4 = a_6-\epsilon_c$	${}^2b_6^6 = a_6+\gamma_c$	${}^2b_6^8 = a_6-\delta$	${}^2b_6^{10} = a_6+\epsilon_c$	${}^2b_6^{12} = a_6-\gamma_c$
${}^2b_7^2 = a_7-\gamma$	${}^2b_7^4 = a_7+\delta_c$	${}^2b_7^6 = a_7-\epsilon$	${}^2b_7^8 = a_7+\gamma$	${}^2b_7^{10} = a_7-\delta_c$	${}^2b_7^{12} = a_7+\epsilon$
${}^2b_8^2 = a_8-\gamma_c$	${}^2b_8^4 = a_8+\delta$	${}^2b_8^6 = a_8-\epsilon_c$	${}^2b_8^8 = a_8+\gamma_c$	${}^2b_8^{10} = a_8-\delta$	${}^2b_8^{12} = a_8+\epsilon_c$
${}^2b_9^2 = a_9+\epsilon$	${}^2b_9^4 = a_9-\gamma$	${}^2b_9^6 = a_9+\delta_c$	${}^2b_9^8 = a_9-\epsilon$	${}^2b_9^{10} = a_9+\gamma$	${}^2b_9^{12} = a_9-\delta_c$
${}^2b_{10}^2 = a_{10}+\epsilon_c$	${}^2b_{10}^4 = a_{10}-\gamma_c$	${}^2b_{10}^6 = a_{10}+\delta$	${}^2b_{10}^8 = a_{10}-\epsilon_c$	${}^2b_{10}^{10} = a_{10}+\gamma_c$	${}^2b_{10}^{12} = a_{10}-\delta$
${}^2b_{11}^2 = a_{11}-\delta_c$	${}^2b_{11}^4 = a_{11}+\epsilon$	${}^2b_{11}^6 = a_{11}-\gamma$	${}^2b_{11}^8 = a_{11}+\delta_c$	${}^2b_{11}^{10} = a_{11}-\epsilon$	${}^2b_{11}^{12} = a_{11}+\gamma$
${}^2b_{12}^2 = a_{12}-\delta$	${}^2b_{12}^4 = a_{12}+\epsilon_c$	${}^2b_{12}^6 = a_{12}-\gamma_c$	${}^2b_{12}^8 = a_{12}+\delta$	${}^2b_{12}^{10} = a_{12}-\epsilon_c$	${}^2b_{12}^{12} = a_{12}+\gamma_c$

As in the case of $l = 1$, it may be seen that all the relevent equations and properties, including (3) (*iii*), (*iv*) §4 and the formation of a group, are satisfied in this case also. The permutation group corresponding to (7.6) can, in this case, be seen to be an intransitive group. Further, the non-associative systems $\bar{S}_2, \bar{S}_4, \ldots, \bar{S}_{12}$ derived, as before, from $\bar{\Sigma}_2, \bar{\Sigma}_4, \ldots, \bar{\Sigma}_{12}$ respectively have the christoffel symbol as their unique elements.

We have seen in (2) §4 that each of the sequences (7.7) is generated by its first term with the help of the pair of operations $(^-, ^+)$ satisfying the property (1) §4. From the nature of these sequences, the pair of operations may be described as follows: Let

$$c = L^t_{ij}, \quad d = \Omega^t_{ij}, \quad \lambda = R^t_{ij} \text{ and } c = d+\lambda,$$

where c, d are coefficients of affine connections and λ is a tensor.

If we write

$$c^* = d^*+\lambda_1 \quad \text{and} \quad c' = d'+\lambda_2$$

then

$$c^- = d^*+\lambda_2 \quad \text{and} \quad c^+ = d'+\lambda_1 \tag{7.8}$$

so that

$$\tfrac{1}{2}(c^*+c') = \tfrac{1}{2}(c^++c^-).$$

The pair of operations are thus described by (7.8).

Finally, the following table shows the 12 sets of ordered sequences, the sets being given in vertical columns:

$$
\begin{array}{cccccccccccc}
l=1 & l=2 & l=3 & l=4 & l=5 & l=6 & l=7 & l=8 & l=9 & l=10 & l=11 & l=12 \\
\Sigma_1 & \Sigma_1 & \Sigma_1 & \Sigma_1 & \Sigma_1 & \Sigma_1 & \Sigma_1 & \Sigma_1 & \Sigma_1 & \Sigma_1 & \Sigma_1 & \Sigma_1 \\
\Sigma_2 & \Sigma_2 & \Sigma_4 & \bar{\Sigma}_4 & \Sigma_6 & \bar{\Sigma}_6 & \Sigma_8 & \bar{\Sigma}_8 & \Sigma_{10} & \bar{\Sigma}_{10} & \Sigma_{12} & \bar{\Sigma}_{13} \\
\Sigma_3 & \Sigma_3 & \Sigma_3 & \Sigma_3 & \Sigma_3 & \Sigma_3 & \Sigma_3 & \Sigma_3 & \Sigma_3 & \Sigma_3 & \Sigma_3 & \Sigma_3 \\
\Sigma_4 & \bar{\Sigma}_4 & \Sigma_6 & \bar{\Sigma}_6 & \Sigma_8 & \bar{\Sigma}_8 & \Sigma_{10} & \bar{\Sigma}_{10} & \Sigma_{12} & \bar{\Sigma}_{12} & \Sigma_2 & \bar{\Sigma}_2 \\
\Sigma_5 & \Sigma_5 & \Sigma_5 & \Sigma_5 & \Sigma_5 & \Sigma_5 & \Sigma_5 & \Sigma_5 & \Sigma_5 & \Sigma_5 & \Sigma_5 & \Sigma_5 \\
\Sigma_6 & \bar{\Sigma}_6 & \Sigma_8 & \bar{\Sigma}_8 & \Sigma_{10} & \Sigma_{10} & \Sigma_{12} & \bar{\Sigma}_{12} & \Sigma_2 & \Sigma_2 & \Sigma_4 & \bar{\Sigma}_4 \\
\Sigma_7 & \Sigma_7 & \Sigma_7 & \Sigma_7 & \Sigma_7 & \Sigma_7 & \Sigma_7 & \Sigma_7 & \Sigma_7 & \Sigma_7 & \Sigma_7 & \Sigma_7 \\
\Sigma_8 & \bar{\Sigma}_8 & \Sigma_{10} & \bar{\Sigma}_{10} & \Sigma_{12} & \bar{\Sigma}_{12} & \Sigma_2 & \bar{\Sigma}_2 & \Sigma_4 & \bar{\Sigma}_4 & \Sigma_6 & \bar{\Sigma}_6 \\
\Sigma_9 & \Sigma_9 & \Sigma_9 & \Sigma_9 & \Sigma_9 & \Sigma_9 & \Sigma_9 & \Sigma_9 & \Sigma_9 & \Sigma_9 & \Sigma_9 & \Sigma_9 \\
\Sigma_{10} & \bar{\Sigma}_{10} & \Sigma_{12} & \bar{\Sigma}_{12} & \Sigma_2 & \Sigma_2 & \Sigma_4 & \bar{\Sigma}_4 & \Sigma_6 & \bar{\Sigma}_6 & \Sigma_8 & \Sigma_8 \\
\Sigma_{11} & \Sigma_{11} & \Sigma_{11} & \Sigma_{11} & \Sigma_{11} & \Sigma_{11} & \Sigma_{11} & \Sigma_{11} & \Sigma_{11} & \Sigma_{11} & \Sigma_{11} & \Sigma_{11} \\
\Sigma_{12} & \bar{\Sigma}_{12} & \Sigma_2 & \Sigma_2 & \Sigma_4 & \Sigma_4 & \Sigma_6 & \bar{\Sigma}_6 & \Sigma_8 & \Sigma_8 & \Sigma_{10} & \Sigma_{10}
\end{array}
\tag{7.9}
$$

The groups formed by the sequences of the sets as well as by the non-associative systems derived from them for the cases $l = 2, \ldots, 12$ can now be obtained from (7.5) and (7.9) and their representations by permutation groups can be obtained from (7.6) and (7.9) by proper substitutions Thus, we obtain groups of parallelism in Riemannian geometry for which (1) and (2) of §5 hold.

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ANHARMONIC PULSATIONS OF ROCHE-MODEL

By

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1. It is well known that the velocity-time curve for cepheid variables is not sinusoidal, in fact the time interval between the maximum velocity of approach and the maximum velocity of recession is nearly five times larger than the interval between the maximum velocity of recession and the maximum velosity of approach. The theory of simple linear harmonic pulsations fails to explain this aspect of velocity time curve. In his George Darwin lecture on the pulsation theory of cepheid-variables Professor Rosseland (1943) discarded the assumption of sinusoidal oscillations usually made in the theoritical investigations and developed the general theory of anharmonic pulsations. Subsequently Bhatnagar and Kothari (1944) obtained for the homogenous model period and skewness for any value of amplitude taking the ratio of specific heats $\gamma = \frac{5}{3}$ and C. Prasad (1949) studied the an-harmonic oscillations of homogenenous model, inverse square model and standard model retaining second order terms in the equation of motion.

In the present note we have studied the anharmonic pulsations of the Roche-model consisting of a central mass point surrounded by an envelope in which the density decreases inversely as the square of the distance from the centre, and of which the mass is so small that the gravity at any point of the model is due to the central mass alone. This model represents the final step in the series of models with increasing mass concentration, homogeneous model being the other extreme case in which there is no mass concentration at all.

2. *Equations of the problem*. Assuming the oscillations essentially adiabatic, the pulsation equation for finite amplitude is

$$\rho_0 r_0 \ddot{\eta} = -(1+\eta)^2 \frac{\partial}{\partial r_0}\left[P_0(1+\eta)^{-2\gamma}(1+\eta+r_0\eta')^{-\gamma}\right] - g_0\rho_0(1+\eta)^{-2} \tag{1}$$

where P_0, ρ_0 and r_0 are the equilibrium values of pressure, density and the distance from the centre, γ the ratio of specific heats and η the relative displacement δ_r/r_0; dashes denote the differentiation with respect to r_0.

For vanishing amplitude equation (1) admits a set of harmonic solutions with periods $2\pi/\sigma_k$ and amplitudes $\eta_k(r_0)$ and for finite amplitude as suggested by Rosseland (1949) we may expand η interms of η_k's and write

$$\eta = \sum_k \eta_k(r_0) q_k(t) \tag{2}$$

Where q_k's are unknown functions of time which are to be determined to know the displacement at any point (r_0) at any time t. We shall normalise η_k's to unity at the surface of the star so that the displacement at any time at the surface will be given by

$$\eta = \sum_k q_k(t). \tag{3}$$

We substitute the value of η from (2) in (1) and expand the right hand side in powers of q_k's. We shall consider the oscillations so small that terms involving powers of q's higher than the second may be neglected. Multiplying by $r_0{}^3\eta_k$ and integrating from the centre $r_0 = 0$ to the boundary $r_0 = R$ of the star we obtain from the orthogonality of η_k's

$$\frac{d^2q_k}{d\tau^2} + \beta_k q_k = \frac{\gamma}{\sigma_1{}^2 I_k}\left[\sum_i D_{iik}q_i^2 + 2\sum_{ij} D_{ijk}q_iq_j\right], \tag{4}$$

where

$$\tau = \sigma_1 t, \tag{5}$$

$$\beta_k = \frac{\sigma_k{}^2}{\sigma_1{}^2}, \tag{6}$$

$$I_k = \int_0^R \rho_0 \gamma_0^4 \eta_k^2 dr_0, \tag{7}$$

and

$$D_{ijk} = -\tfrac{1}{2}\left(3 - \frac{4}{\gamma}\right)(3\gamma+1)\int_0^R P_0' r_0^3 \eta_i\eta_j\eta_k dr_0 + \tfrac{1}{2}(3\gamma-1)\int_0^R P_0 r_0^4 (\eta_i\eta_j'\eta_k' + \eta_i'\eta_j\eta_k'$$

$$+ \eta_i'\eta_j'\eta_k) dr_0 + \tfrac{1}{2}(\gamma+1)\int_0^R P_0 r_0^5 \eta_i'\eta_j'\eta_k' dr_0, \tag{8}$$

and i, j, and k take all positive integral values. These equations are general for all models. τ is the new variable chosen to make the period of the fundamental mode equal to 2π for vanishing amplitude.

The periodic solutions of the equations (4) are required having the same period for all modes. Therefore we write

$$\left.\begin{aligned}
q_1 &= a_0 + a_1\cos n\tau + a_2\cos 2n\tau + a_3\cos 3n\tau + \cdots \\
q_2 &= b_0 + b_1\cos n\tau + b_2\cos 2n\tau + b_3\cos 3n\tau + \cdots \\
q_3 &= c_0 + c_1\cos n\tau + c_2\cos 2n\tau + c_3\cos 3n\tau + \cdots \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{aligned}\right\} \tag{9}$$

Substituting these values of q_k's in equations (4) and comparing the co-efficients of $\cos kn\tau$ on the two sides we get infinite number of infinite equations for the constants which are solved by successive approximations approaching nearer and nearer to the correct solutions. Thus we get the values of constants $a_0, a_2, a_3 \ldots; b_0, b_1, b_2, b_3, \ldots$ and $c_0, c_1, c_2, c_3, \ldots$ interms of one constant, say a_1, there by determining q_k's The details of the method are given in the appendix.

3. For Roche-model we have, putting $r_0 = xR$

$$g_0 = \frac{GM}{x^2R^2}, \tag{10}$$

$$\rho_0 = \frac{\rho_1}{x^2}, \tag{11}$$

$$P_0 = \int_0^R \frac{\rho_1}{x^2} \frac{GM}{R^2 x^2} d(Rx) = \frac{GM\rho_1}{3R} \frac{1-x^3}{x^3}, \tag{12}$$

where ρ_1 is the value of the density at the surface of the star and other notations have their usual meanings.

η_k's normalised for unity at the surface of the star are given by

$$\left.\begin{aligned}
\eta_1 &= x^q, \\
\eta_2 &= -\frac{2q+3}{3} x^q \left(1 - \frac{2(q+3)}{2q+3} x^3\right), \\
\eta_3 &= \frac{(2q+3)(q+3)}{9} x^q \left(1 - \frac{2(2q+9)}{2q+3} x^3 + \frac{(q+6)(2q+9)}{(q+3)(2q+3)} x^6\right),
\end{aligned}\right\} \tag{13}$$

and in general

$$\eta_{k+1} = \frac{(-1)^k}{3^k \underline{|k}} x^{-q} \left(\frac{d}{x^2 dx}\right)^k \{x^{2q+3k}(1-x^3)^k\},$$

for $k = 0, 1, 2, \ldots$

where

$$q = \sqrt{3\alpha} \quad \text{and} \quad \alpha = 3 - \frac{4}{\gamma}.$$

Substituting the values of P_0, g_0 and ρ_0 from (10), (11) and (12) in equations (4)-(7) we get

$$\ddot{q}_k + \beta_k q_k = \frac{1}{F_1 I_k} \left[\sum_i D_{iik} q_i^2 + 2 \sum_{ij} D_{ijk} q_i q_j\right] \tag{14}$$

where

$$I_k = \int_0^1 x^2 \eta_k^2 dx, \tag{15}$$

$$D_{ijk} = \frac{3}{2}\left(3 - \frac{4}{\gamma}\right)(3\gamma+1) \int_0^1 \eta_i \eta_j \eta_k \frac{dx}{x} + \tfrac{1}{2}(3\gamma-1) \int_0^1 x(1-x^3)(\eta_i \eta_j' \eta_k' + \eta_i' \eta_j \eta_k'$$
$$+ \eta_i' \eta_j' \eta_k) dx + \tfrac{1}{2}(\gamma+1) \int_0^1 x^2(1-x^3) \eta_i' \eta_j' \eta_k' dx, \tag{16}$$

and

$$F_1 = q(q+3), \tag{17}$$

From (6) and (7) we get

$$\left.\begin{aligned}
\beta_1 &= 1, \\
\beta_2 &= \frac{q+6}{q}, \\
\beta_3 &= \frac{(q+6)(q+9)}{q(q+3)}, \\
\beta_{k+1} &= \frac{F_{k+1}}{F_1} = \frac{(q+3k)(q+3k+3)}{q(q+3)},
\end{aligned}\right\} \text{(18)} \quad \text{and} \quad \left.\begin{aligned}
I_1 &= \frac{1}{2q+3}, \\
I_2 &= \frac{1}{2q+9}, \\
I_3 &= \frac{1}{2q+15}, \\
I_{k+1} &= \frac{1}{2q+6k+3},
\end{aligned}\right\} \text{(19)}$$

Consistent with our approximations we consider only first three modes for which the equations (14) may be written as

$$\left.\begin{aligned} \frac{d^2q_1}{d\tau^2}+q_1 &= \frac{1}{F_1I_1}\left[D_{111}q_1^2+2D_{121}q_1q_2+2D_{131}q_1q_3\right], \\ \frac{d^2q_2}{d\tau^2}+\beta_2q_2 &= \frac{1}{F_1I_2}\left[D_{112}q_1^2+2D_{122}q_1q_2+2D_{132}q_1q_3\right], \\ \frac{d^2q_3}{d\tau^2}+\beta_3q_3 &= \frac{1}{F_1I_3}\left[D_{113}q_1^2+2D_{123}q_1q_2+2D_{133}q_1q_3\right], \end{aligned}\right\} \quad (20)$$

The terms containing q_2^2, q_3^2 and q_2q_3 have been neglected as their contributions will be small as will be seen from (22). On evaluating the co-efficients D_{ijk} for $\gamma = \frac{5}{3}$ the equations (20) reduce to

$$\left.\begin{aligned} \ddot{q}_1+q_1 &= 2.76034q_1^2+2\cdot862105q_1q_2+1\cdot02791q_1q_3, \\ \ddot{q}_2+5\cdot472136q_2 &= 2\cdot9420007q_1^2+28\cdot216315q_1q_2+10\cdot775365q_1q_3, \\ \ddot{q}_3+13\cdot03486q_3 &= 1\cdot592227q_1^2+16\cdot309098q_1q_2+66\cdot60948q_1q_3, \end{aligned}\right\} \quad (21)$$

where

$$\ddot{q}_k \equiv \frac{d^2q_k}{d\tau^2}$$

Substituting (9) in (21) we obtain the coefficients $a_0, a_2, \ldots$; $b_0, b_1, b_2, \ldots$; and $c_0, c_1, c_2, \ldots$ as shown in the appendix, in terms of a_1, which in its turn was chosen to be 0.05 which gives the relative mean surface amplitude of oscillation equal to 1/20. This value of a_1 is so chosen that the terms neglected in approximation may not affect the results obtained. If we take much higher valoes of a_1, the approximation will be vitiated. The solutions of (21) are found to be as follows:

$$\left.\begin{aligned} q_1 &= 0.003568+0\cdot05\cos n\tau-0\cdot001205\cos 2n\tau-0\cdot0000047\cos 3n\tau+\cdots \\ q_2 &= 0.0008251+0.0009622\cos n\tau+0.002784\cos 2n\tau-0\cdot0005525\cos 3n\tau+\cdots \\ q_3 &= 0.0002289+0\cdot0003145\cos n\tau+0\cdot0003908\cos 2n\tau+0\cdot0004074\cos 3n\tau\ldots \end{aligned}\right\} \quad (22)$$

where

$$n^2 = 0.9760416.$$

The total displacement at the surface of the star will be

$$\eta(R) = \Sigma q_k = 0\cdot004622+0\cdot0512767\cos n\tau+0.00197\cos 2n\tau-0.0001498\cos 3n\tau\ldots; \quad (23)$$

For mean relative amplitude equal to 1/20, the skewness in the velocity time curve is found to be 1.075 and the increase in the period to be 1.2% of the period for the fundamental mode for vanishing amplitude. For the same amplitude the Rosseland's (1943) approximate treatment for homogeneous model gives period increase of about 1% and the skewness equal to about 1.15

APPENDIX

In the following we give briefly the method of solving equations determining the coefficients in (9). Substituting these values of q_1, q_2 and q_3 in (20) which can be conveniently put in the form

$$\left.\begin{aligned} \ddot{q}_1+q_1 &= A_1q_1^2+2B_1q_1q_2+2C_1q_1q_3, \\ \ddot{q}_2+\beta_2q_2 &= A_2q_1^2+2B_2q_1q_2+2C_2q_1q_3, \\ \ddot{q}_3+\beta_3q_3 &= A_3q_1^2+2B_3q_1q_2+2C_3q_1q_3, \end{aligned}\right\} \tag{20'}$$

and equating the coefficients of $\cos kn\tau (k = 0, 1, 2, 3, \ldots)$ on the two sides of each of the above equations we have

$$\begin{array}{l} a_0 \\ \beta_2 b_0 \\ \beta_3 c_0 \end{array} = \begin{array}{r} A_1 \\ A_2 \\ A_3 \end{array} \left[\begin{array}{l} a_0^2+\frac{1}{2}a_1^2+\frac{1}{2}a_2^2 \\ +\frac{1}{2}a_3^2+\frac{1}{2}a_4^2 \\ \end{array}\right] \begin{array}{r} 2B_1 \\ +2B_2 \\ 2B_3 \end{array} \left[\begin{array}{l} a_0b_0+\frac{1}{2}a_1b_1 \\ +\frac{1}{2}a_2b_2+\frac{1}{2}a_3b_3 \\ +\frac{1}{2}a_4b_4 \end{array}\right] \begin{array}{r} 2C_1 \\ +2C_2 \\ 2C_3 \end{array} \left[\begin{array}{l} a_0c_0+\frac{1}{2}a_1c_1 \\ +\frac{1}{2}a_2c_2+\frac{1}{2}a_3c_3 \\ +\frac{1}{2}a_4c_4 \end{array}\right], \tag{24a}$$

$$\begin{array}{l} (1-n^2)a_1 \\ (\beta_2-n^2)b_1 \\ (\beta_3-n^2)c_1 \end{array} = \begin{array}{r} A_1 \\ A_2 \\ A_3 \end{array} \left[\begin{array}{l} 2a_0a_1+a_1a_2 \\ +a_2a_3+a_3a_4 \\ \end{array}\right] \begin{array}{r} 2B_1 \\ +2B_2 \\ 2B_3 \end{array} \left[\begin{array}{l} a_1b_0+b_1a_0 \\ +\frac{1}{2}(a_1b_2+b_1a_2) \\ +\frac{1}{2}(a_2b_3+b_2a_3) \end{array}\right] \begin{array}{r} 2C_1 \\ +2C_2 \\ 2C_3 \end{array} \left[\begin{array}{l} a_1c_0+a_0c_1 \\ +\frac{1}{2}(a_1c_2+a_2c_1) \\ +\frac{1}{2}(a_2c_3+a_3c_2) \end{array}\right], \tag{24b}$$

$$\begin{array}{l} (1-4n^2)a_2 \\ (\beta_2-4n^2)b_2 \\ (\beta_3-4n^2)c_2 \end{array} = \begin{array}{r} A_1 \\ A_2 \\ A_3 \end{array} \left[\begin{array}{l} \frac{1}{2}a_1^2+2a_0a_2 \\ +a_1a_3+a_2a_4 \\ \end{array}\right] \begin{array}{r} 2B_1 \\ +2B_2 \\ 2B_3 \end{array} \left[\begin{array}{l} \frac{1}{2}a_1b_1+a_0b_2+b_0a_2 \\ +\frac{1}{2}a_1b_3+\frac{1}{2}a_3b_1 \\ +\frac{1}{2}a_2b_4+\frac{1}{2}a_4b_2 \end{array}\right] \begin{array}{r} 2C_1 \\ +2C_2 \\ 2C_3 \end{array} \left[\begin{array}{l} \frac{1}{2}a_1c_1+a_0c_2 \\ +a_2c_0+\frac{1}{2}a_1c_3 \\ +\frac{1}{2}a_3c_1+\frac{1}{2}a_2c_4 \\ +\frac{1}{2}a_4c_2 \end{array}\right], \tag{24c}$$

$$\begin{array}{l} (1-9n^2)a_3 \\ (\beta_2-9n^2)b_3 \\ (\beta_3-9n^2)c_3 \end{array} = \begin{array}{r} A_1 \\ A_2 \\ A_3 \end{array} \left[\begin{array}{l} a_1a_2+a_0a_3 \\ +a_1a_4 \\ \end{array}\right] \begin{array}{r} 2B_1 \\ +2B_2 \\ 2B_3 \end{array} \left[\begin{array}{l} \frac{1}{2}a_1b_2+\frac{1}{2}a_2b_1 \\ +a_0b_3+a_3b_0 \\ +\frac{1}{2}a_1b_4+\frac{1}{2}a_4b_1 \end{array}\right] \begin{array}{r} 2C_1 \\ +2C_2 \\ 2C_3 \end{array} \left[\begin{array}{l} \frac{1}{2}a_1c_2+\frac{1}{2}a_2c_1 \\ +a_0c_3+c_0a_3 \\ +\frac{1}{2}a_1c_4+\frac{1}{2}c_1a_4 \end{array}\right], \tag{24d}$$

where terms in the same horizontal line are to be considered together except the expressions within square brackets which are same in each case.

To get the first approximation for a_0, a_2; b_0, b_2 and c_0, c_2 interms of a_1 we neglect on the right hand side of (24) in the expressions within the square brackets all terms other than in a_1 only, giving us

$$a_0 = \tfrac{1}{2}A_1a_1^2, \qquad b_0 = \frac{1}{2}\frac{A_2}{\beta_2}a_1^2, \qquad c_0 = \frac{1}{2}\frac{A_3}{\beta_3}a_1^2,$$

$$a_2 = -\tfrac{1}{6}A_1a_1^2, \quad b_2 = \frac{1}{2}\frac{A_2}{\beta_2-4}a_1^2, \quad c_0 = \frac{1}{2}\frac{A_3}{\beta_3-4}a_1^2,$$

To evalute the first order approximations for n^2, a_3; b_1, b_3; c_1 and c_3 we substitute these values and $n = 1$ in the required equations. We have taken $n = 1$ as is suggested by first equation of (20′) the coefficient of q_1 being unity on the left hand side. We obtained these co-efficients as follows:

$$b_1 = \frac{1}{(\beta_2-1)}\left\{\frac{5A_1A_2}{6}+\frac{A_2B_2(3\beta_2-8)}{2\beta_2(\beta_2-4)}+\frac{A_3C_2(3\beta_3-8)}{2\beta_3(\beta_3-4)}\right\}a_1^2,$$

$$c_1 = \frac{1}{(\beta_3-1)}\left\{\frac{5A_1A_3}{6}+\frac{A_2B_3(3\beta_2-8)}{2\beta_2(\beta_2-4)}+\frac{A_3C_3(3\beta_3-8)}{2\beta_3(\beta_3-4)}\right\}a_1^2,$$

$$n^2 = 1-\left\{\frac{5A_1^2}{6}+\frac{A_2B_1(3\beta_2-8)}{2\beta_2(\beta_2-4)}+\frac{A_3C_1(3\beta_3-8)}{2\beta_3(\beta_3-4)}\right\}a_1^2,$$

$$a_3 = \left\{\frac{A_1^2}{3}-\frac{A_2B_1}{\beta_2-4}-\frac{A_3C_1}{\beta_3-4}\right\}\frac{a_1^3}{16},$$

$$b_3 = \left\{-\frac{A_1A_2}{3}+\frac{A_2B_2}{\beta_2-4}+\frac{A_3C_2}{\beta_3-4}\right\}\frac{a_1^3}{2(\beta_2-9)},$$

$$c_3 = \left\{-\frac{A_1A_3}{3}+\frac{A_2B_3}{\beta_2-4}+\frac{A_3C_3}{\beta_3-4}\right\}\frac{a_1^3}{2(\beta_3-9)}.$$

To get the second approximation for these coefficients we substitute the first order values in the right hand side of each of the equations (24) retaining only terms upto a_3, b_3 and c_3. This process was repeated a number of times till we got the values of these coefficients to the desired degree of accuracy. In this note we repeated this process four times to ensure occurcy upto five places of decimals.

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NOTE ON A CIRCULAR CUBIC WITH A REAL COINCIDENCE POINT AT INFINITY

By

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Introduction

This short paper deals primarily with a circular cubic Γ, having its real point (K) at infinity for a *coincidence point* and only secondarily with the *more general type* of cubic (circular or otherwise), having, of course, K for a coincidence point. Although for the most part only the *bicursal*† type of cubic (with a *real* coincidence point at infinity) is contemplated, still a good many of the results, obtained in this paper, will, when properly modified, hold also for the *unicursal* cubic.

As a matter of convenience the paper has been divided into two sections. Sec. I disposes of the subject geometrically by aid of the Theory of Residuation. Then Sec. II treats of the subject analytically by using *homogeneous* as well as *Cartesian* systems of coordinates. Incidentally we have had to make use of the traditional properties of the mixed invariants Θ, Θ', attached to a *pair* of conics.

For the sake of brevity a *coincidence point* (of a plane cubic Γ) has been contracted as a "c-point". Throughout this paper the symbols I, J have been used to denote the two *circular points* at infinity and the symbol K stands for the third point (obviously real) where Γ is met by the *line at infinity.*

For the sake of clearness, we re-state* here the main result (marked F in Art. 5):—

If a circular cubic Γ partakes of any one of the following properties, viz.

(i) *that its real point at infinity should be a c-point,*

(ii) *that one of its cyclic points should be a c-point,*

(iii) *that one of its cyclic points should lie on the real asymptote,*

(iv) *that one of its centres of inversion should be a c-point,*

and (v) *that the osculating circle at a centre of inversion should pass through a cyclic point,*

then Γ must partake of all the remaining properties. Furthermore a circular cubic Γ, endowed with one (and, therefore, with all) of the five properties, can, by an appropriate choice of rectangular Cartesian axes, be thrown into the form:

$$x\{(x^2+y^2)+\lambda(\mu^2-1)x-2\lambda\mu y\} = k(x-\lambda)(y-\mu x),$$

where k, λ, μ are constants

† As is well-known, a cubic—or, for the matter of that, any algebraic plane curve—is said to be *bicursal* or *unicursal*, according as its *deficiency* (or *genus*) is 1 or 0.

* It is at the wise suggestion of the learned referee that Prop. F (Art. 5) has been re-stated in the *Introduction*.

Although for the purpose of *rapprochement* we have at times felt constrained to touch on *known* results, still the major portion of the results, embodied in this paper, is believed to be new. At any rate, we are not aware whether the problem in its present aspect has attracted much attention heretofore.

Section I

(Geometrical Investigation based on the Theory of Residuation)

1. Suppose (Fig. 1) that a (bicursal) circular cubic Γ has its real point (K) at infinity for a c-point (*i.e.*, a coincidence point). If then I, J be the two circular points at infinity (lying of course on Γ), we have the two equations of residuation:

$$[I+J+K] = 0 \quad \text{and} \quad [9K] = 0. \qquad (1), (2)$$

Supposing further that α and A are respectively the first and second tangentials of K we have the two additional relations, *viz.*,

$$[2K+\alpha] = 0 \quad \text{and} \quad [2\alpha+A] = 0. \qquad (3), (4)$$

Remarking that the four equations (1), (2), (3), (4), taken together, automaticlly lead to the subsidiary relations:

$$[9\alpha] = 0, \quad [9A] = 0 \quad \text{and} \quad [2A+K] = 0, \qquad (5), (6), (7)$$

we conclude that α and A also are c-points and that K is the tangential of A. Then K coincides with its own third tangential—a fact, otherwise implied in the very assumption that K is a c-point. Needless to say, similar properties hold also for the other two c-points α and A. In other words $K\alpha A$ counts as one among the twenty-four H-triangles (*i.e.*, Hart-triangles) of the cubic. In view of its intrinsic importance, $K\alpha A$ will be designated in this paper as the "principal" H-triangle of Γ.

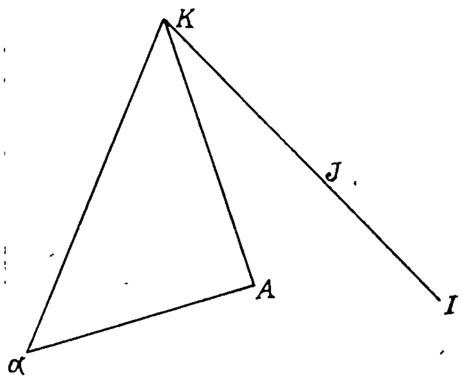

Fig. 1.

Inasmuch as the combination of (1), (2), (3), (4) gives rise to another subsidiary relation, *viz.*,

$$[4\alpha+I+J] = 0, \qquad (8)$$

it appears that the c-point α, which is obviously the point, where the *real* asymptote of Γ meets the cubic again, counts as one among the sixteen cyclic points of Γ. Accordingly α will be designated as the "principal" cyclic point of Γ. Moreover the c-point A, which is easily seen, by (7), to be one of the four centres of inversion, will be designated as the "principal" centre of inversion, possessing the distinctive property that it is the tangential of the "principal" cyclic point (or, what is the same thing) the second tangential of the real point K at infinity.

Suppose now that B is *any one* of the four centres of inversion (taken at random) and that B' is the point, where the osculating circle at B cuts Γ again. We then have the equations of residuation:

$$[2B+K] = 0 \quad \text{and} \quad [3B+B'+I+J] = 0. \tag{9), (10}$$

Because (9), (10), combined with (1), (2), give:

$$[4B'+I+J] = 0, \tag{11}$$

it follows that B' is a cyclic point. Thus for a circular cubic Γ, restricted by the condition (2), the osculating circle at *any one* of the four centres of inversion (say B) meets Γ at a cyclic point B'. In the particular case when B becomes the *"principal"* centre of inversion, B' is easily verified to coincide with the *"principal"* cyclic point α. For, the equations (1), (2), (3) and (4) automatically lead to:

$$[3A+\alpha+I+J] = 0,$$

proving the point at issue.

The results obtained in this article may then be summed up in the form of a proposition.—

Prop. A—*If a bicursal circular cubic Γ has its real point K at infinity for a c point, then:*

(i) *one of the cyclic points must be a c-point,*

(ii) *one of the cyclic points must lie on the real asymptote,*

(iii) *one of the centres of inversion must be a c-point,*

and (iv) *the circle of curvature of Γ at any of its (four) centres of inversion must pass through a cyclic point.*

In the next article we shall examine how far the converse of the afore-mentioned properties is valid.

2. There are several cases to consider.

Case I. Firstly suppose that one of the cyclic points (say α) is a c-point. Then we start with the equations (1), (5) and (8) and thence easliy deduce (2) and (3).

We then arrive at the proposition:—

Prop. B—*If one of the cyclic points (say, α) of a circular cubic Γ be a c-point then the real point (K) of Γ at infinity must also be a c-point and α itself will lie on the real asymptote of the cubic.*

Case II. Secondly, suppose that one of the centres of inversion (say, A) is a c-point.

Then we start with (6), (7) and deduce (2) immediately.

We thus establish the proposition:—

Prop. C—*If one of the centres of inversion of a circular cubic be a c-point, then the real point (K) of Γ at infinity must also be a c-point.*

Case III. Thirdly, suppose that the osculating circle of a circular cubic Γ at a centre of inversion (B) passes through a cyclic point (B'). Then at the very outset we have the relations (9), (10), (11), besides (1). Plainly this combination of equations leads in the long run to (2). We thus arrive at the proposition.—

Prop. D—*If the osculating circle of a circular cubic Γ at a centre of inversion goes through a cyclic point, the real point K (on Γ) an infinity must be a c-point.*

The converse propositions, corresponding to different phases of Prop. A being thus disposed of by the geometrical theory of residuation, we propose to devote Sec. II to the *analytical* aspect, and to form *homogeneous* as well as *Cartesian* equations of the general as well as the special type of cubic, having a real c-point at infinity.

Section II

(Analytical method based on the use of homogeneous and Cartesian coordinates)

8. Suppose (Fig. 2) that Γ is an unrestricted cubic (circular or otherwise), having the triangle of reference for one of its H-triangles. If the tangentials of A, B, C be B, C, A respectively, the equation to Γ is known* to be

$$px^2z + qy^2x + rz^2y + sxyz = 0, \tag{12}$$

where $p:q:r:s$ are three effective parameters.

Manifestly (12) can be alternatively put in each of the following symbolic forms:

$$Sx = -rz^2y, \quad S'y = -px^2z, \quad S''z = -qy^2x, \qquad (13), (14), (15)$$

where the three conics S, S', S'' are defined by:

$$\left.\begin{aligned} S &\equiv qy^2 + syz + pzx = 0, \\ S' &\equiv rz^2 + szx + qxy = 0, \\ \text{and} \quad S'' &\equiv px^2 + sxy + ryz = 0, \end{aligned}\right\} \qquad (1)$$

Observing that the three right lines $z = 0$, $x = 0$, $y = 0$ touch the three conics S, S', S'' respectively at the points A, B, C and interpreting geometrically the

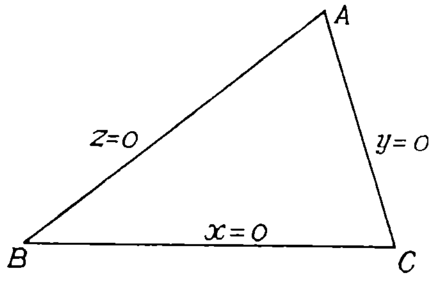

Fig 2

homogeneous equations (13), (14), (15) of Γ, we readily realise that S, S', S'' are respectively the osculating conics of Γ at A, B, C.

* To be given three correlated c-points, *viz.*, the vertices of an H-triangle is evidently tantamount to 3×2 or 6 conditions. So it is meet and proper that the typical equation of the cubic should involve $(9-6)$ or 3 *arbitrary* constants. It is hardly necessary to remark that if in the H-triangle the tangentials of A, B, C were respectively C, A, B insted of B, C, A, the equation to Γ would assume the form:

$$p'xz^2 + q'yx^2 + r'zy^2 + s'xyz = 0.$$

[Cf : Hilton (1920), p. 236, Ex. 2].

Remarking that the mixed invariant Θ', attaching to the conic-pair (S, S') is *nil* and referring to *known* lemmas*, we infer that *there must exist a triangle, self-conjugate to S but circumscribed about S'*. Similar relations must from symmetry subsist between the conics (S', S'') and between (S'', S).

Evidently the polar conics U, U', U'' of the three vertices (A, B, C) of the H-triangle $A\,B\,C$ *w. r. t.* the cubic Γ are respectively:

$$\left.\begin{aligned} U &\equiv qy^2+syz+2pzx = 0, \\ U' &\equiv rz^2+szx+2qxy = 0 \\ \text{and} \quad U'' &\equiv px^2+sxy+2ryz = 0. \end{aligned}\right\} \tag{II}$$

The mixed invariant Θ, attaching to the conic-pair (U, U') being easily seen to vanish, we conclude, on referring once again to the afore-mentioned articles of Salmon and Askwith, that *there must exist a triangle. self-conjugate to U but inscribed in U'*. Symmetry shews that similar relations subsist between (U', U'') and between (U'', U)

The results obtained in this article may then be summed up in the form of a substantive proposition :—

Prop. E—*If a bicursal cubic Γ (circular or otherwise) has ABC for one of its (twentyfour) Hart-triangles, such that the tangentials† of A, B, C, are respectively B, C, A, and if (S, S', S'') be the three osculating conics and (U, U', U'') the three polar conics, answering to the points A, B, C, then*:

(i) *there must exist a Δ, self-conjugate w. r. t. S but circumcribed about S',*
,, ,, ,, ,, ,, S' ,, ,, ,, S'',
,, ,, ,, ,, ,, S'' ,, ,, ,, S;

and

(ii) *there must exist a Δ, self-conjugate w. r. t. U but inscribed in U',*
,, ,, ,, ,, ,, U' ,, U'',
,, ,, ,, ,, ,, U'' ,, U.

4. Having disposed of the more general type of cubic in the foregoing article, we shall now revert to the consideration of the special variety of cubic, discussed in Sec. I. Retaining our previous notations and conventions, we may choose the "principal" H-triangle $K\alpha A$ (Fig. 8) as the triangle of reference. For the sake of precision, suppose that the *homogeneous* equations of the three sides $K\alpha$, αA and AK of the triangle of reference are respectively:

* Vide Salmon (1911), § 375, p. 340 and Askwith (1935), § 370, p. 389.

† It is hardly necessary to add that in the *alternative* case when the tangentials of A, B, C are respectively C, A, B, instead of B, C, A, the two conics of each of the pairs $(S\,S'')$, $(S''S)$, (SS'), $(U'U'')$, $(U''U)$, and (UU') are to change places, in order that the geometrical relations, mentioned in (i) and (ii), may hold good.

$$X = 0, \quad Y = 0 \quad \text{and} \quad Z = 0.$$

Then the *homogeneous* equation to Γ must necessarily be of the type:

$$aX^2Z + bY^2X + cZ^2Y + dXYZ = 0, \tag{16}$$

where $a : b : c : d$ are parameters.

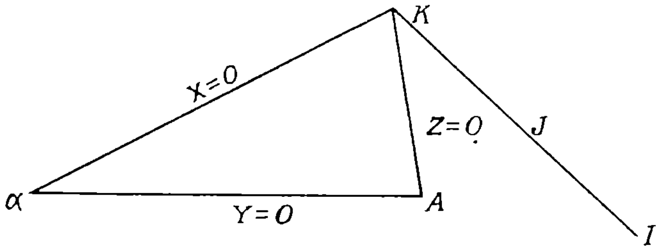

Fig. 3.

If we now choose α as the origin of rectangular Cartesian axes, the real asymptote (αK) as the y-axis, so that the x-axis is the line through α perpendicular to αK, it is clear that the Cartesian equation of αA must be of the form. $y = \mu x$. The multiplicative constants, inherently present in the structure of the homogeneous coordinates X, Y, Z, can evidently be so adjusted that the mutual relations between the two sets of coordinates (X, Y, Z) and (x, y) may be representable in the forms.

$$X = x, \quad Z = y - \mu x \quad \text{and} \quad Z = x - \lambda. \tag{17}$$

Substitution of (17) in (16) at once gives the Cartesian equation of Γ in the form:

$$ax^2(x-\lambda) + bx(y-\mu x)^2 + c(x-\lambda)^2(y-\mu x) + dx(x-\lambda)(y-\mu x) = 0. \tag{18}$$

For (18) to define a *circular* cubic it is essential that the set of homogeneous cubic terms on the L.S. of (18), *viz.*

$$ax^3 + bx(y-\mu x)^2 + cx^2(y-\mu x) + dx^2(y-\mu x)$$

should involve each of the two linear factors $y \pm ix$, $(i \equiv \sqrt{-1})$.

The requisite conditions being easily seen to be:

$$a = b(1+\mu^2) \quad \text{and} \quad c + d = 2\mu b, \tag{19}$$

the Cartesian equation of Γ can without much difficulty be thrown into the symbolic form:

$$xS = k(x-\lambda)(y-\mu x), \tag{20}$$

where S is a circle and k a constant, given by:

$$S \equiv x^2 + y^2 + \lambda(\mu^2-1)x - 2\lambda\mu y \quad \text{and} \quad k \equiv \frac{c\lambda}{b}. \tag{21}$$

Regard being had to the tangency of the line $(x = \lambda)$ with the circle $(S = 0)$ at the "principal" centre of inversion A $(\lambda, \lambda\mu)$, we easily find, on interpreting the equation (20) geometrically, that the circle S osculates the cubic Γ at A (Fig 4). Evidently this result is quite in consonance with (iv) of Prop. A. It is worth mentioning that, whereas, generally speaking, the osculating circle at any of the four centres of inversion goes

through *some* cyclic point of Γ, the osculating circle at the "principal"[1] centre of inversion goes through the "principal" cyclic point of the curve.

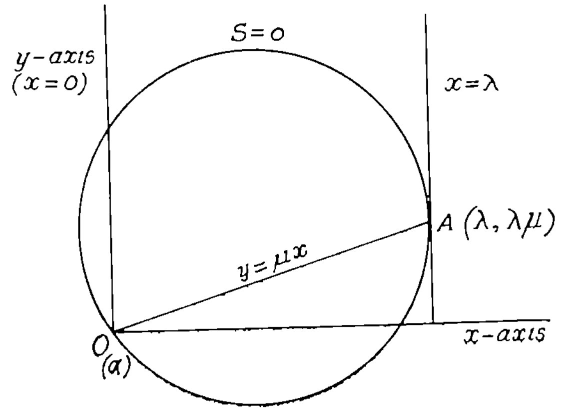

Fig. 4

If the transformation-scheme (17) be kept in view, one can readily deduce from (I) and (II):

(*a*) that the osculating conics (S, S', S'') of Γ answering to the "principal" centre of inversion A, the real point (K) of Γ at infinity and the "principal" cyclic point (α) are representable in the Cartesian forms:

$$S \equiv b(y-\mu x)^2 + d(x-\lambda)(y-\mu x) + ax(x-\lambda) = 0, \tag{22}$$

$$S' \equiv c(x-\lambda)^2 + dx(x-\lambda) + bx(y-\mu x) = 0, \tag{23}$$

and

$$S'' \equiv ax^2 + dx(y-\mu x) + c(x-\lambda)(y-\mu x) = 0, \tag{24}$$

and (*b*) that the polar conics (U, U', U'') of the same three points are representable in the Cartesian forms:

$$U \equiv b(y-\mu x)^2 + d(x-\lambda)(y-\mu x) + 2ax(x-\lambda) = 0, \tag{25}$$

$$U' \equiv c(x-\lambda)^2 + dx(x-\lambda) + 2bx(y-\mu x) = 0, \tag{26}$$

and

$$U'' \equiv ax^2 + dx(y-\mu a) + 2c(x-\lambda)(y-\mu x) = 0. \tag{27}$$

Interested readers may at pleasure utilise the six Cartesian equations (22)-(27) with the attached conditions (19) so as to investigate further geometrical relations among (S, S', S'') and (U, U', U''), other than those already mentioned in Prop. E.

5. We shall now close this topic by recapitulating some of the results of this section along with those of the previous section. In fact, if we amalgamate Props. A, B, C, D and attend to the results of Art. 4, we may finalise our conclusions in the following form.—

Prop. F—*If a circular cubic Γ partakes of any one of the following properties, viz.,*

(*i*) *that its real point at infinity should be a c-point,*

(*ii*) *that one of its cyclic points should be a c-point,*

(*iii*) *that one of its cyclic points should lie on the real asymptote,*

(*iv*) *that one of its centres of inversion should be a c-point,*

and (v) that the osculating circle at a centre of inversion should pass through a cyclic point, then it (Γ) *must partake of all the remaining properties. Furthermore, a circular cubic* Γ *endowed with one (and, therefore, with all) of the five properties can, by an appropriate choice of rectangular Cartesian axes, be thrown into the form :*

$$x\{x^2+y^2+\lambda(\mu^2-1)x-2\lambda\mu y\} = k(x-\lambda)(y-\mu x),$$

where k, λ, μ *are constants.*

References

Askwith, (1935), *Analytical Geometry of the Conic Sections.*
Hilton, (1920), *Plane Algebraic Curves.*
Salmon, (1911), *Treatise on Conic Sections.*

A THEOREM CONCERNING AN ASYMPTOTIC INTEGRAL

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Let E be a linear measurable set contained in (a, b). Let $\varphi(x)$ and all the functions of $\{f_n(x)\}$ be integrable in Lebesgue sense over the set E. Under suitable conditions we shall give in this note an asymptotic evaluation of the integral

$$I_n = \int_E \varphi(x) f_1(x) f_2(x) \ldots f_n(x) dx, \tag{1}$$

as $n \to \infty$. The result will include Laplace-Darboux's asymptotic formula, Beppo Levi's extension (Levi, 1946) and the generalization mentioned in Pólya-Szegö's *Analysis* (Pólya-Szegö, 1925) as special cases. We shall need the following

Definition. *A sequence of functions $\{f_n(x)\}$ are said to attain their positive absolute maximum uniformly at $x = \xi$, if for any given $d > 0$, there is a $\delta > 0$ (independent of n) such that, for all n,*

$$|f_n(x)| \leq f_n(\xi) - \delta, \quad \text{whenever } |x - \xi| \geq d. \tag{2}$$

An asymptotic evaluation of I_n is contained in the following

Theorem. *Let $\varphi(x), f_1(x), f_2(x), \ldots, f_n(x), \ldots$ $(a \leq x \leq b)$ be uniformly bounded real-valued functions Lebesgue integrable over $E(E \subset (a, b))$ such that*

(i) *$\{f_n(x)\}$ attain their positive absolute maximum uniformly at an interior point ξ of E,*

(ii) *ξ belongs to both the Lebesgue set for the characteristic function of E and the Lebesgue set for $\varphi(x)$ with $\varphi(x) \neq 0$,*

(iii) *there exists a constant $\lambda > 0$ such that*

$$\lim_{x \to \xi} |f_n(x) - f_n(\xi)| / |x - \xi|^\lambda = k_n, \quad (0 < k_n < \infty) \tag{3}$$

where $\underline{\lim}\, k_n > 0$, $\overline{\lim}\, k_n < \infty$, and all the limits k_n's are attained uniformly with regard to n as $x \to \xi$. Then for n large we have

$$I_n \sim \frac{2\Gamma(1/\lambda)\varphi(\xi) f_1(\xi) f_2(\xi) \ldots f_n(\xi)}{\lambda\{(k_1/f_1(\xi)) + (k_2/f_2(\xi)) + \cdots + (k_n/f_n(\xi))\}^{1/\lambda}}. \tag{4}$$

If the second half of hypothesis (ii) concerning $\varphi(x)$ is replaced by

(ii)′ *both $\varphi(\xi-)$ and $\varphi(\xi+)$ exist and do not vanish,*

then as a corollary of the theorem we have the following asymptotic expression

$$I_n \sim \frac{\Gamma(1/\lambda)(\varphi(\xi-) + \varphi(\xi+)) f_1(\xi) f_2(\xi) \ldots f_n(\xi)}{\lambda\{(k_1/f_1(\xi)) + (k_2/f_2(\xi)) + \cdots + (k_n/f_n(\xi))\}^{1/\lambda}}. \tag{5}$$

The classical Laplace's asymptotic formula (Widder, 1940) can easily be deduced from (5) by assuming $f_n(x) = f(x)$, $\lambda = 2$, $k_n = k = -\frac{1}{2}f''(\xi) > 0$, $\varphi(\xi-) = \varphi(\xi+) = \varphi(\xi)$, $E = (a, b)$. Similarly, B. Levi's formula is a special case of (5) with $f_n(x) = f(x)$, $k_n = k > 0$, $\varphi(\xi-) = \varphi(\xi+) = \varphi(\xi)$, $E = (a, b)$.

Proof of the Theorem. Here we shall generalize the method of Laplace and Widder (1946). Note that $\varphi(x)f_1(x)\ldots f_n(x)/(f_1(\xi)\ldots f_n(\xi)) \leq \varphi(x)$ where the dominating function $|\varphi(x)|$ is integrable. So clearly $\varphi(x)f_1(x)\ldots f_n(x)$ is integrable for every n. Let $E_0 = E \cap (a, \xi)$, $E_1 = E \cap (\xi, b)$, so that $E = E_0 \cup E_1$. Write

$$A_n = \sum_{j=1}^{n}(k_j/f_j(\xi)). \tag{6}$$

It suffices to consider the integral

$$J_n = A_n^{1/\lambda}\int_{E_1}\varphi(x)\left(\frac{f_1(x)\ldots f_n(x)}{f_1(\xi)\ldots f_n(\xi)}\right)dx. \tag{7}$$

For, the case where E_1 is replaced by E_0 can be treated exactly in the same way. Using hypothesis (i), (iii) and applying logarithmic expansion we easily find that (within a certain neighbourhood of ξ, say $|x-\xi|<\Delta$)

$$\log(f_n(x)/f_n(\xi)) = -k_n|x-\xi|^\lambda(1+\theta_n(x, \xi))/f_n(\xi), \tag{8}$$

where the fact that $\theta_n(x, \xi)\to 0$ uniformly with respect to n as $x\to\xi$ is easily justified by noticing that $\lim k_n > 0$, $\lim k_n < \infty$, bound $f_n(\xi) > 0$. Consequently we have

$$\frac{f_1(x)\ldots f_n(x)}{f_1(\xi)\ldots f_n(\xi)} = \exp\left[-|x-\xi|^\lambda\sum_{j=1}^{n}(k_j/f_j(\xi))(1+\theta_j)\right], \text{ where } \theta_j(x, \xi) = \theta_j. \tag{9}$$

Since $\theta_n\to 0$ uniformly as $x\to\xi$, and $\overline{\lim}\, f_n(\xi)<\infty$, it follows that

$$\sum_{j=1}^{\infty}(k_j/f_j(\xi))(1+\theta_j(x, \xi)) = \infty, \tag{10}$$

provided that x is sufficiently near ξ. This fact permits us to transform a dominating part of J_n into Eulerian integral by making $n\to\infty$, so that the asymptotic value of J_n can be determined therefrom. More precisely, for an arbitrary $\epsilon>0$ we have to choose a small interval with length $d>0$, say $(\xi, \xi+d) = U$, so that J_n can be expressed as

$$J_n = \left(\int_{E_1\cap U}+\int_{E_1-(E_1\cap U)}\right)A_n^{1/\lambda}\varphi(x)\exp\left[-|x-\xi|^\lambda\sum_{j=1}^{n}(k_j/f_j(\xi))(1+\theta_j(x, \xi))\right]dx$$

$$= \int_{E_1\cap U}A_n^{1/\lambda}\varphi(\xi)\exp[\cdots]dx+\int_{E_1\cap U}A_n^{1/\lambda}[\varphi(x)-\varphi(\xi)]\exp[\cdots]dx$$

$$+\int_{E_1-(E_1\cap U)}A_n^{1/\lambda}\varphi(x)\exp[\cdots]\,dx = L_1+L_2+L_3.$$

Further, in terms of the characteristic function $\chi(x)$ of E, we may express

$$L_1 = \int_\xi^{\xi+d}A_n^{1/\lambda}\varphi(\xi)\chi(x)\exp[\ldots]dx$$

$$= \int_\xi^{\xi+d}A_n^{1/\lambda}\varphi(\xi)\exp[\ldots]dx+\int_\xi^{\xi+d}A_n^{1/\lambda}\varphi(\xi)[\chi(x)-\chi(\xi)]\exp[\ldots]\,dx$$

$$= L_4+L_5,$$

where $\chi(\xi) = 1$, since ξ is a point of E.

Note that $A_n^{1/\lambda}\exp[\ldots]\to 0$ as $n\to\infty$, for all x in $E_1-(E_1\cap U)$. Thus clearly for any fixed small $d=d(\epsilon)>0$ we can always determine $N=N(d)>0$ so large that $|L_3|<\epsilon$ whenever $n>N$. We shall now show that $L_2=o(1)$, $L_5=o(1)$ for large n and small d. By hypothesis (ii) we may denote

$$\psi(x)=\int_\xi^x|\varphi(u)-\varphi(\xi)|\,du=o(|x-\xi|)\quad(x\to\xi),\tag{11}$$

so that $\psi(\xi)=0$. For given $\epsilon(0<\epsilon<\frac{1}{2})$ we may always choose $d>0$ so small that

$$|\theta_n(x,\xi)|<\epsilon,\quad|\psi(x)|<\epsilon|x-\xi|\quad\text{whenever}\quad|x-\xi|\leq d,\ n=1,2,3,\ldots\tag{12}$$

Thus integration by parts gives

$$|L_2|\leq\int_\xi^{\xi+\xi}A_n^{1/\lambda}|\varphi(x)-\varphi(\xi)|\exp\Big[-|x-\xi|^\lambda\sum_{j=1}^n(k_j/f_j(\xi))(1-\epsilon)\Big]dx$$

$$=\psi(\xi+d)A_n^{1/\lambda}\exp[-d^\lambda(1-\epsilon)A_n]-\int_\xi^{\xi+d}A_n^{1/\lambda}\psi(x)\left\{\frac{d}{dx}\exp[-(1-\epsilon)A_n(x-\xi)^\lambda]\right\}dx$$

$$=o(1)-\lambda(1-\epsilon)A_n^{1+1/\lambda}\int_\xi^{\xi+d}\psi(x)(x-\xi)^{\lambda-1}\exp[-(1-\epsilon)A_n(x-\xi)^\lambda]dx,\tag{13}$$

where the term $o(1)(n\to\infty)$ is implied by $A_n^{1/\lambda}\exp[-d^\lambda(1-\epsilon)A_n]\to 0$. Now using (12) and making the substitution

$$(1-\epsilon)A_n(x-\xi)^\lambda=t^\lambda,\quad(x\geq\xi)\tag{14}$$

we easily find that the second term of (13) is less than

$$\epsilon\lambda(1-\epsilon)A_n^{1+1/\lambda}\int_\xi^{\xi+d}(x-\xi)^\lambda\exp[-(1-\epsilon)A_n(x-\xi)^\lambda]dx$$

$$\leq\epsilon\lambda(1-\epsilon)^{-1/\lambda}\int_0^\infty t^\lambda e^{-t^\lambda}dt=O(\epsilon).$$

This shows that $|L_2|$ can be made as small as we like, provided that d is chosen to be sufficiently small and n is allowed to tend to infinity. Moreover, by exactly the same treatment we easily find that $L_5=o(1)$, since L_5 is similar to L_2 and hypothsis (ii) implies

$$\int_\xi^x|\chi(u)-\chi(\xi)|\,du=o(|x-\xi|)\quad(x\to\xi).\tag{15}$$

Hence the dominating part of J_n should be L_4. In fact, by the uniformity of $\theta_n\to 0(x\to\xi)$ and by the transformation

$$A_n(x-\xi)^\lambda=s,\quad(x\geq\xi)\tag{16}$$

it can easily be shown that, for $n\to\infty$,

$$L_4\sim\frac{1}{\lambda}\varphi(\xi)\int_0^{A_nd^\lambda}e^{-s}s^{1/\lambda-1}ds\sim\frac{1}{\lambda}\varphi(\xi)\Gamma(1/\lambda),\tag{17}$$

since $A_nd^\lambda\to\infty$ as $n\to\infty$. Now since $J_n\sim L_4$ it follows that

$$\int_{E_1} \varphi(x) f_1(x) \ldots f_n(x) dx \sim \frac{1}{\lambda} A_n^{-1/\lambda} \Gamma(1/\lambda) \varphi(\xi) f_1(\xi) \ldots f_n(\xi). \tag{18}$$

Similarly we may obtain the same asymptotic expression for $\int_{E_2}$. Hence we have the formula (4).

Remark. If $H = (\xi - d, \xi + d)$ and if $\mu(E \cap H)$ denotes the Lebesgue measure of $E \cap H$, then it is immediately deduced from (15) that

$$\lim_{d \to 0} \frac{\mu(E \cap H)}{2d} = 1. \tag{19}$$

In other words, the density (*Cf.* Titchmarsh, 1932) of E at ξ is unity. Conversely, if ξ is any point of E at which the density is 1, then ξ must belong to the Lebesgue set for the characteristic function $\chi(x)$. Hence the first part of hypothesis (ii) may also be replaced by

(ii)* *E has unity density at* $x = \xi$.

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ON THE CONVERGENCE AND SUMMABILITY—(C, 1) OF AN ANALOGOUS CONJUGATE FOURIER SERIES

By

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Mitra (1949) has shown, that if we have a sequence

$$\{a_n(x)\} = \frac{1}{2}\int_0^{2\pi} f(t)\cos[n\pi \sin \tfrac{1}{2}(t-x)]dt$$

then the series $\frac{1}{2}a_0(x)+\sum_{n=1}^{\infty} a_n(x)$, henceforward called the *Analogous Fourier series*, converges to $f(x)$ under the same conditions as the Fourier series; he further showed that the conditions for the series to be summable-$(C, 1)$ to $f(x)$ are also the same in both the cases.

The object of the present paper is to prove that if we have a sequence defined by

$$\{b_n(x)\} = -\frac{1}{2}\int_0^{2\pi} f(t)\sin[n\pi \sin \tfrac{1}{2}(t-x)]dt$$

then the series $\sum_{n=1}^{\infty} b_n(x)$, henceforward called the *Analogous Conjugate Fourier series*, converges to

$$\bar{f}(x) = -\int_0^{\pi} \frac{\psi^*(t)}{2tg\frac{1}{2}t}\cdot\frac{dt}{\sqrt{(\pi^2-t^2)}}$$

under the same conditions as the conjugate Fourier series; further, that the conditions for the conjugate series to be summable-$(C, 1)$ to $\bar{f}(x)$ are also the same in both the cases.

It is interesting to note here that $\bar{f}(x)$ is not exactly the same for the two conjugate series.

I. We may also note here that

$$b_z(x) = -\frac{1}{2}\int_0^{2\pi} f(t)\sin[z\pi \sin \tfrac{1}{2}(t-x)]dt$$

is defined for all real values of z. Since (Watson, 1922, p. 22)

$$\sin[z\pi \sin \tfrac{1}{2}(t-x)] = 2\sum_{n=0}^{\infty} J_{2n+1}(z\pi)\sin \tfrac{1}{2}(2n+1)(t-x);$$

we have on multiplying both sides by $f(t)$ and integrating term-by-term

$$b_z(x) = \pi\sum_{n=0}^{\infty} J_{2n+1}(z\pi)\bar{b}_n(x) \qquad (A)$$

where

$$\bar{b}_n(x) = -\frac{1}{\pi}\int_0^{2\pi} f(t)\sin \tfrac{1}{2}(2n+1)(t-x)dt,$$

provided that term-by-term integration is permissible.

Again since, (Watson, 1922, p. 533)

$$\int_0^\infty J_{2m}(z)J_{2n}(z)z^{-1}dz = 0 \quad (m \neq n)$$

$$= \frac{1}{4n} \quad (m = n)$$

we have on multiplying (A) by $J_{2n+1}(z\pi)z^{-1}$ and integrating term-by-term

$$\int_0^\infty b_z(x)J_{2n+1}(z\pi)z^{-1}dz = \frac{\pi}{4n+2}\bar{b}_n(x) \tag{B}$$

The relations (*A*) and (*B*) show that, given the sequence $\{b_n(x)\}$ we can construct the sequence $\{\bar{b}_n(x)\}$; and vice-versa.

II. *Convergence of the Analogous Fourier series and its conjugate series.*

If

$$\left.\begin{aligned} a_n(x) &= \frac{1}{2}\int_0^{2\pi} f(t)\cos[n\pi\sin\tfrac{1}{2}(t-x)]dt \\ b_n(x) &= -\frac{1}{2}\int_0^{2\pi} f(t)\sin[n\pi\sin\tfrac{1}{2}(t-x)]dt \\ \text{and} \qquad f(x\pm 2\pi) &= f(x) \end{aligned}\right\} \tag{1}$$

and if

$$\mathfrak{S}[f] \equiv \tfrac{1}{2}a_0 + \sum_{n=1}^\infty a_n(x); \qquad \bar{\mathfrak{S}}[f] \equiv \sum_{n=1}^\infty b_n(x) \tag{2}$$

then the n-th partial sums $S_n(x) \equiv S_n(x; f)$ and $\bar{S}_n(x) \equiv \bar{S}_n(x; f)$ respectively of the series $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ can be written as:

$$\left.\begin{aligned} S_n(x) &= \frac{1}{2}\int_0^{2\pi} f(t+x)A_n(t)dt \\ \text{and} \qquad \bar{S}_n(x) &= -\frac{1}{2}\int_0^{2\pi} f(t+x)\bar{A}_n(t)dt \end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned} A_n(t) &= \frac{\sin[(n+\frac{1}{2})\pi\sin\frac{1}{2}t]}{2\sin[\frac{1}{2}\pi\sin\frac{1}{2}t]} \\ \text{and} \qquad \bar{A}_n(t) &= \frac{\cos[\frac{1}{2}\pi\sin\frac{1}{2}t]-\cos[(n+\frac{1}{2})\pi\sin\frac{1}{2}t]}{2\sin[\frac{1}{2}\pi\sin\frac{1}{2}t]} \end{aligned}\right\}. \tag{4}$$

The functions A_n and $\bar{A}_n$ will be called the *Analogous Dirichlet Kernel* and the *Analogous Dirichlet Conjugate Kernel* respectively.

Let

$$S_n^*(x) = S_n(x) - \tfrac{1}{2}a_n; \quad \bar{S}_n^*(x) = \bar{S}_n(x) - \tfrac{1}{2}b_n; \tag{5}$$

and we have

$$\left.\begin{aligned} S_n^*(x) &= \frac{1}{2}\int_0^{2\pi} f(x+t)A_n^*(t)dt \\ \text{and}\quad \bar{S}_n^*(x) &= -\frac{1}{2}\int_0^{2\pi} f(x+t)\bar{A}_n^*(t)dt \end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned} A_n^*(t) &= \frac{\sin[n\pi\sin\frac{1}{2}t]}{2\mathrm{tg}[\frac{1}{2}\pi\sin\frac{1}{2}t]} \\ \text{and}\quad \bar{A}_n^*(t) &= \frac{1-\cos[n\pi\sin\frac{1}{2}t]}{2\mathrm{tg}[\frac{1}{2}\pi\sin\frac{1}{2}t]} \end{aligned}\right\}. \tag{7}$$

If $f \equiv 1$, then (Watson, 1922, p. 633).

$$\sigma_n^* = \sigma_n - \frac{1}{4}\int_0^{2\pi}\cos[n\pi\sin\tfrac{1}{2}t]dt = \tfrac{1}{2}\pi + \pi\sum_1^n J_0(n\pi) - \tfrac{1}{2}\pi J_0(n\pi)$$

$$\therefore \quad \text{as } n \to \infty, \quad \mathrm{Lt}\ \sigma_n^* \to 1$$

Putting $I_n^* = S_n^*(x) - \sigma_n^* f(x)$ and $\bar{I}_n^* = \bar{S}_n^*(x)$ and observing that $A_n^*(t)$ is even while $\bar{A}_n^*(t)$ is odd we have

$$\left.\begin{aligned} I_n^* &= \frac{1}{2}\int_0^{\pi}\frac{\varphi(t)}{2\mathrm{tg}[\frac{1}{2}\pi\sin\frac{1}{2}t]}\sin[n\pi\sin\tfrac{1}{2}t]dt \\ \text{and}\quad \bar{I}_n^* &= -\frac{1}{2}\int_0^{\pi}\frac{\psi(t)}{2\mathrm{tg}[\frac{1}{2}\pi\sin\frac{1}{2}t]}\{1-\cos[n\pi\sin\tfrac{1}{2}t]\}dt \end{aligned}\right\} \tag{8}$$

where $\varphi(t)$ and $\psi(t)$ have their usual meanings, *viz.*,

$$\varphi(t) = f(x+t)+f(x-t)-2f(x) \quad \text{and} \quad \psi(t) = f(x+t)-f(x-t).$$

Putting $\pi\sin\frac{1}{2}t = u$, we have

$$\left.\begin{aligned} I_n^* &= \int_0^{\pi}\frac{\varphi^*(t)\sin nt}{2\mathrm{tg}\frac{1}{2}t\ \sqrt{(\pi^2-t^2)}}\,dt \\ \text{and}\quad \bar{I}_n^* &= -\int_0^{\pi}\frac{\psi^*(t)(1-\cos nt)}{2\mathrm{tg}\frac{1}{2}t.\sqrt{(\pi^2-t^2)}}\,dt \end{aligned}\right\} \tag{9}$$

where $\varphi^*(u) = \varphi(t)$ and $\psi^*(u) = \psi(t)$.

Let

$$\left.\begin{aligned} J_n^* &= \frac{1}{\pi}\int_0^{\pi}\frac{\sin nt}{2\mathrm{tg}\frac{1}{2}t}\cdot\varphi^*(t)dt \\ \text{and}\quad \bar{J}_n^* &= -\frac{1}{\pi}\int_0^{\pi}\frac{1-\cos nt}{2\mathrm{tg}\frac{1}{2}t}\cdot\psi^*(t)dt \end{aligned}\right\} \tag{10}$$

then

$$\left.\begin{aligned} I_n^*-J_n^* &= \int_0^{\pi}\frac{\sin nt}{2\mathrm{tg}\frac{1}{2}t}\varphi^*(t)\left\{\frac{1}{\sqrt{(\pi^2-t^2)}}-\frac{1}{\pi}\right\}dt \\ \text{and}\quad \bar{I}_n^*-\bar{J}_n^* &= -\int_0^{\pi}\frac{1-\cos nt}{2\mathrm{tg}\frac{1}{2}t}\psi^*(t)\left\{\frac{1}{\sqrt{(\pi^2-t^2)}}-\frac{1}{\pi}\right\} \end{aligned}\right\} \tag{11}$$

Since
$$\frac{\varphi^*(t)}{2\mathrm{tg}\frac{1}{2}t}\left\{\frac{1}{\sqrt{(\pi^2-t^2)}}-\frac{1}{\pi}\right\} \quad \text{and} \quad \frac{\psi^*(t)}{2\mathrm{tg}\frac{1}{2}t}\left\{\frac{1}{\sqrt{(\pi^2-t^2)}}-\frac{1}{\pi}\right\}$$
are summable in $(0, \pi)$ it follows by the Riemann-Lebesgue Theorem that as $n \to \infty$,
$$I_n^* - J_n^* \to 0$$
and
$$\bar{I}_n^* - \bar{J}_n^* \to -\int_0^\pi \frac{\psi^*(t)}{2\mathrm{tg}\frac{1}{2}t}\left\{\frac{1}{\sqrt{(\pi^2-t^2)}}-\frac{1}{\pi}\right\}dt$$
on the sole assumption that $f(x)$ is summable in $(0, \pi)$ and is periodic. Hence, as $n \to \infty$
$$S_n^*(x) \to f(x) \text{ if } J_n^* \to 0 \tag{12}$$
and
$$\bar{S}_n^*(x) \to -\int_0^\pi \frac{\psi^*(t)}{2\mathrm{tg}\frac{1}{2}t}\cdot\frac{dt}{\sqrt{(\pi^2-t^2)}}$$
if
$$\bar{J}_n^* \to -\frac{1}{\pi}\int_0^\pi \frac{\psi^*(t)dt}{2\mathrm{tg}\frac{1}{2}t}.$$

We see that J_n^* and $\bar{J}_n^*$ do not at all differ respectively from S_n^* and $\bar{S}_n^*$, the corresponding partial sums of the Fourier series of $f(x)$ and its conjugate series.

From the relation connecting t and u $[u = \pi \sin \frac{1}{2}t]$ all the usual conditions for S_n^* and $\bar{S}_n^*$ will be applicable to J_n^* and $\bar{J}_n^*$ respectively and vice-versa.

Hence we have:

The Analogous Dini's Test: *If the first of the integrals*
$$\int_0^\pi \frac{|\varphi^*(t)|}{2tg\frac{1}{2}t}dt, \qquad \int_0^\pi \frac{|\psi^*(t)|}{2tg\frac{1}{2}t}dt, \tag{13}$$
is finite, then $\mathfrak{S}[f]$ *converges at* x *to the sum* $f(x)$. *If the second integral is finite,* $\bar{\mathfrak{S}}[f]$ *converges at the point* x *to the value which we shall denote by* $\bar{f}(x)$, *where*
$$\bar{f}(x) = -\int_0^\pi \frac{\psi^*(t)}{2\mathrm{tg}\frac{1}{2}t}\cdot\frac{dt}{\sqrt{(\pi^2-t^2)}}.$$

For the proof we notice that $S_n^*(x) - \sigma_n^* f(x)$ and $\bar{S}_n^*(x) - \bar{f}(x)$, by Eqs. (10), are respectively Fourier sine and cosine coefficients of integrable functions.

III. *Summability-(C, 1) of Analogous Fourier Series.*

From Mitra (1949, §III) we know that if
$$C_n(t) = (A_0 + A_1 + \cdots + A_{n-1})/n$$
then
$$nC_n(t) = \tfrac{1}{2}(\sin[n\tfrac{1}{2}\pi \sin \tfrac{1}{2}t]/\sin[\tfrac{1}{2}\pi \sin \tfrac{1}{2}t])^2.$$

The expression $C_n(t)$, henceforward called the *Analogous Fejer's Kernel*, satisfies all the three properties of positive kernels and basing our proof exactly on the lines of Zygmund (1935) it is easy to prove*:

* As the summability $-(C, 1)$ of the Analogous Fourier series has previously been dealt with by Mitra (1949, §III), a bare outline of the proof is given here. This proof is different from that given by him and leads us to prove the Analogous Fejer's Theorem and the corollaries. All this can easily be verified.

The 'Analogous' Fejér's Theorem: If the limits $f(x\pm 0)$ exist, $\mathfrak{S}[f]$, i.e. $\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n(x)$, is summable-(C, 1) at the point x to the value $\frac{1}{2}[f(x+0)+f(x-0)]$, in particular, if f is continuous at x, $\mathfrak{S}[f]$ is summable there to the value $f(x)$. If f is continuous at every point of an internal $I=<a, b>$, $\mathfrak{S}[f]$ is uniformly summable in I.

From the Analogous Fejer's theorem all the usual corollaries, which are true in the case of Fourier series, can be easily derived for the Analogous Fourier series,

IV. *Summability $-(C, 1)$ of the Analogous Conjugate Fourier Series.*

Let

$$\bar{\sigma}_n=(S_0+S_1+\cdots+S_{n-1})/n, \text{ then}$$

$$\bar{\sigma}_n=-\frac{1}{4n}\int_0^{\pi} \frac{(n-1)\cos\left[\frac{1}{2}\pi \sin\frac{1}{2}t\right]-\cos\left[\frac{3}{2}\pi\sin\frac{1}{2}t\right]-\cdots-\cos\left[\frac{1}{2}\overline{2n-1}\pi\sin\frac{1}{2}t\right]}{\sin\left[\frac{1}{2}\pi\sin\frac{1}{2}t\right]}\psi(t)dt$$

$$=-\frac{1}{4n}\int_0^{\pi}\frac{n\sin\left[\pi\sin\frac{1}{2}t\right]-\sin\left[n\pi\sin\frac{1}{2}t\right]}{2\sin^2\left[\frac{1}{2}\pi\sin\frac{1}{2}t\right]}\psi(t)dt$$

i.e.

$$\bar{\sigma}_n+\frac{1}{4}\int_0^{\pi}\frac{\psi(t)}{\operatorname{tg}\left[\frac{1}{2}\pi\sin\frac{1}{2}t\right]}dt=\frac{1}{4n}\int_0^{\pi}\frac{\sin\left[n\pi\sin\frac{1}{2}t\right]}{2\sin^2\left[\frac{1}{2}\pi\sin\frac{1}{2}t\right]}\psi(t)dt$$

Let $\pi\sin\frac{1}{2}t=v$ and evaluating as in §II:

$$\bar{I}_n=\bar{\sigma}_n+\frac{1}{2}\int_0^{\pi}\frac{\psi^*(t)}{\operatorname{tg}\frac{1}{2}t}\cdot\frac{dt}{\sqrt{(\pi^2-t^2)}}=\frac{1}{2n}\int_0^{\pi}\frac{\sin nt}{2\sin^2\frac{1}{2}t}\cdot\frac{\psi^*(t)}{\sqrt{(\pi^2-t^2)}}\,dt$$

Putting

$$\bar{J}_n=\frac{1}{2\pi n}\int_0^{\pi}\frac{\sin nt}{2\sin^2\frac{1}{2}t}\psi^*(t)dt$$

we have

$$\bar{I}_n-\bar{J}_n=\frac{1}{2n}\int_0^{\pi}\frac{\sin nt}{2\sin^2\frac{1}{2}t}\psi^*(t)\left[\frac{1}{\sqrt{(\pi^2-t^2)}}-\frac{1}{\pi}\right]dt.$$

Therefore

$$|\bar{I}_n-\bar{J}_n|\leq\frac{1}{2n}\int_0^{\pi}\frac{1}{2\sin^2\frac{1}{2}t}\psi^*(t)\left[\frac{1}{\sqrt{(\pi^2-t^2)}}-\frac{1}{\pi}\right]dt$$

and as the integral on the right exists we have

$$\bar{I}_n-\bar{J}_n\to 0 \text{ as } n\to\infty.$$

Hence the condition for $\mathfrak{S}[f]$ to be summable-(C, 1) to the sum

$$\bar{f}(x)=-\int_0^{\pi}\frac{\psi^*(t)}{2\operatorname{tg}\frac{1}{2}t}\cdot\frac{dt}{\sqrt{(\pi^2-t^2)}}$$

becomes

$$\operatorname*{Lt}_{n\to\infty}\frac{1}{n}\int_0^{\pi}\frac{\sin nt}{\sin^2\frac{1}{2}t}\psi^*(t)\,dt=0.$$

Hence all the theorems on (C, 1)-summability of Conjugate Fourier Series follow easily.

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SOME SIMPLE PROBLEMS OF THICK CONICAL SHELLS

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Introduction

The object of this paper is to find stresses in a thick shell bounded by co-axial conical surfaces due to an axial force and a couple about the axis applied at the common vertex. The problem of a solid cone under an axial force and a couple were solved by Michell (1900) and Ghosh (1936) respectively. A different but more direct method is employed in this paper to find some simple results which are believed to be new.

1. *Stresses in a conical shell due to an axial force.*

Let u, v, w stand for components u_r, u_θ, u_ϕ in polar co-ordinates. Then assuming that w is zero and u, v are independent of the azimuthal angle ϕ, we get the equations of equilibrium under no body forces as,

$$\left.\begin{aligned}(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{\sin\theta}\frac{\partial}{\partial\theta}(\varpi r\sin\theta)=0\\ (\lambda+2\mu)\sin\theta\frac{\partial\Delta}{\partial\theta}-2\mu\frac{\partial}{\partial r}(\varpi r\sin\theta)=0\end{aligned}\right\}\tag{1}$$

where,

$$\Delta=\frac{\partial u}{\partial r}+\frac{2u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{v}{r}\cot\theta\tag{2}$$

$$2\varpi=\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta}.\tag{3}$$

Let

$$u=\frac{f_1(\theta)}{r}\text{ and }v=\frac{f_2(\theta)}{r},\tag{4}$$

r being the distance from the common vertex. Then from (2), (3) and (4) we get,

$$2\varpi=-\frac{f'_1(\theta)}{r^2}\tag{5}$$

$$\Delta=\frac{f_1(\theta)}{r^2}+\frac{f'_2(\theta)}{r^2}+\frac{f_2(\theta)}{r^2}\cot\theta\tag{6}$$

where dashes denote differentiation with respect to θ.

It is clear that the equations (1) are satisfied if we take,

$$\Delta=-\frac{kD_1\cos\theta}{r^2}\tag{7}$$

and

$$2\varpi = -\frac{D_1 \sin\theta}{r^2} \tag{8}$$

where

$$k = \frac{\mu}{\lambda+2\mu} \tag{9}$$

and D_1 is a constant.

From (5) and (8) we get,

$$f_1(\theta) = D_1 \cos\theta + D_0 \tag{10}$$

where D_0 is a constant.

Substituting the expression for $f_1(\theta)$ and Δ in (6) we get,

$$f'_2(\theta)+f_2(\theta)\cot\theta = -(k+1)D_1\cos\theta - D_0.$$

or,

$$\frac{d}{d\theta}\{\sin\theta\, f_2(\theta)\} = -\left(\frac{k+1}{2}\right)\sin 2\theta - D_0\sin\theta$$

which gives

$$f_2(\theta) = D_0\cot\theta + E_0\operatorname{cosec}\theta - \frac{k+1}{2}D_1\sin\theta \tag{11}$$

E_0 being a constant. Hence finally we get,

$$u = \frac{1}{r}(D_0 + D_1\cos\theta) \tag{12}$$

$$v = \frac{1}{r}\left(D_0\cot\theta + E_0\operatorname{cosec}\theta - \frac{k+1}{2}D_1\sin\theta\right) \tag{13}$$

The components of stress given by the displacements (12) and (13) are,

$$\left.\begin{aligned}
\widehat{rr} &= \lambda\Delta + 2\mu\frac{\partial u}{\partial r} = \frac{1}{r^2}\{-(3-2k)\mu D_1\cos\theta - 2\mu D_0\}\\
\widehat{\theta\theta} &= \lambda\Delta + 2\mu\left(\frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r}\right) = \frac{1}{r^2}\{k\mu D_1\cos\theta - 2\mu D_0\cot^2\theta - 2\mu E_0\operatorname{cosec}\theta\cot\theta\}\\
\widehat{\phi\phi} &= \lambda\Delta + 2\mu\left(\frac{v}{r}\cot\theta + \frac{u}{r}\right) = \frac{1}{r^2}\{k\mu D_1\cos\theta + 2\mu D_0\operatorname{cosec}^2\theta + 2\mu E_0\operatorname{cosec}\theta\cot\theta\}\\
\widehat{r\theta} &= \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right) = \frac{1}{r^2}\{k\mu D_1\sin\theta - 2\mu D_0\cot\theta - 2\mu E_0\operatorname{cosec}\theta\}\\
\widehat{\theta\phi} &= \widehat{r\phi} = 0,
\end{aligned}\right\} \tag{14}$$

The boundary conditions are that $\widehat{\theta\theta}$ and $\widehat{r\theta}$ are to vanish simultaneously when $\theta=\alpha$ and $\theta=\beta$ since $\widehat{\theta\phi}=0$. We see from (14) that both the relations are satisfied if,

$$kD_1\sin\theta - 2D_0\cot\theta - 2E_0\operatorname{cosec}\theta = 0 \text{ when } \theta=\alpha \text{ and } \theta=\beta.$$

Hence,

$$\left.\begin{aligned}
kD_1\sin\alpha - 2D_0\cot\alpha - 2E_0\operatorname{cosec}\alpha &= 0\\
kD_1\sin\beta - 2D_0\cot\beta - 2E_0\operatorname{cosec}\beta &= 0
\end{aligned}\right\}$$

whence we get

$$\left.\begin{aligned} 2D_0 &= -kD_1(\cos\alpha+\cos\beta) \\ 2E_0 &= kD_1(1+\cos\alpha\cos\beta) \end{aligned}\right\} \tag{15}$$

So, out of three constants we express D_0 and E_0 in terms of D_1. The constant D_1 will be obtained if the magnitude of the force applied along the axis of the shell is given.

If X, Y, Z, L, M, N be the components of the resultant traction across the portion of a small sphere of radius r bounded by the cone, the centre of the sphere being at the vertex of the conical shell,

$$\left.\begin{aligned} X &= \iint(\widehat{rr}\sin\theta\cos\phi+\widehat{r\theta}\cos\theta\cos\phi)r^2\sin\theta\,d\theta d\phi, \\ Y &= \iint(\widehat{rr}\sin\theta\sin\phi+\widehat{r\theta}\cos\theta\cos\phi)r^2\sin\theta\,d\theta d\phi, \\ Z &= \iint(\widehat{rr}\cos\theta-\widehat{r\theta}\sin\theta)r^2\sin\theta\,d\theta d\phi, \\ L &= \iint(-\widehat{r\theta}\,r\sin\phi)r^2\sin\theta\,d\theta d\phi, \\ M &= \iint(\widehat{r\theta}\,r\cos\phi)r^2\sin\theta\,d\theta d\phi, \\ N &= 0. \end{aligned}\right\} \tag{16}$$

since $\widehat{r\phi}=\widehat{\theta\phi}=0$.

From (14), (15) and (16) we get,

$$X=Y=L=M=N=0.$$

$$\begin{aligned} Z = \int_0^{2\pi}\int_\beta^\beta \{ &-(3-2k)\mu D_1\sin\theta\cos^2\theta-2\mu D_0\sin\theta\cos\theta \\ &-k\mu D_1\sin^3\theta+2\mu D_0\sin\theta\cos\theta+2\mu E_0\sin\theta\}d\theta d\phi \\ &= 2\pi kD_1\{\lambda(\cos^3\beta-\cos^3\alpha)+\mu(\cos^2\beta+\cos^2\alpha)(\cos\beta-\cos\alpha)\}. \end{aligned}$$

If we put $\alpha=0$. Michell's result is obtained, *viz.*

$$Z=\frac{2\pi\mu D_1}{\lambda+2\mu}\{\lambda(\cos^3\beta-1)+\mu(\cos^2\beta+1)(\cos\beta-1)\},$$

the case of a solid cone under axial force.

2. Stresses due to an axial couple at the vertex.

We consider now $u=v=0$ and w independent of ϕ. Then $\Delta=0$. The components of rotation are,

$$2\varpi_r=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(wr\sin\theta) \tag{1'}$$

$$2\varpi_\theta = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(wr\sin\theta). \tag{2}$$

The equation of equilibrium is therefore,

$$\frac{\partial}{\partial r}(r\varpi_\theta) - \frac{\partial\varpi_r}{\partial\theta} = 0$$

or,

$$r^2\frac{\partial^2 w}{\partial r^2} + 2r\frac{\partial w}{\partial r} + \frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\theta} + w\cot\theta\right) = 0 \tag{3'}$$

in absence of body forces.

Let

$$w = \frac{f(\theta)}{r^2}. \tag{4'}$$

Then from (3′) we get,

$$f''(\theta) + f'(\theta)\cot\theta - f(\theta)(\operatorname{cosec}^2\theta - 2) = 0.$$

It is evident that $f(\theta) = A\sin\theta$ is a solution where A is a constant. Therefore,

$$w = \frac{A\sin\theta}{r^2}. \tag{5'}$$

The stresses $\widehat{r\theta} = \widehat{\theta\theta} = \widehat{\theta\phi} = \widehat{rr} = \widehat{\phi\phi} = 0$. Boundaries $\theta = \alpha$ and $\theta = \beta$ are threfore free from tractions.

$$\widehat{r\phi} = \mu\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right) = -\frac{3A\mu\sin\theta}{r^3}. \tag{6'}$$

As in (16) of the previous section, $X = Y = Z = L = M = 0$. Constant A is determined when N is given.

$$N = \int_0^{2\pi}\int_\alpha^\beta -\frac{3A\mu\sin\theta}{r^3}\cdot\; r\sin\theta\, r^2\sin\theta\, d\theta d\phi = 2\pi\mu A\{3(\cos\beta - \cos\alpha) - (\cos^3\beta - \cos^3\alpha)\}.$$

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ON THE BENDING OF AN ELASTIC PLATE—II

By

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The author determined the deflexion of the central plane of a thin elliptic plate made of isotropic elastic material, clamped at the edge carrying a load of weight W concentrated at any point of its upper face (Sengupta, 1949). This note gives expression for the deflexion of the central plane of (*i*) a thin semi-elliptic plate clamped at the elliptic edge and supported along the major axis, (*ii*) a thin semi-elliptic plate clamped at the elliptic edge and supported along the minor axis, and (*iii*) a quadrant of a thin elliptic plate clamped at the elliptic edge and supported along the straight edges, each carrying a load of weight W concentrated at any point of the upper face of the plate.

Let the bounding ellipse be given by the equation

$$x^2/a^2+y^2/b^2 = 1, \quad (a > b > 0)$$

Introducing elliptic co-ordinates by the transformation $x+iy = c\cosh(\xi+i\eta)$, $(c > 0)$, $\xi = \alpha$, $(\alpha > 0)$ represents the bounding ellipse, provided $a = c\cosh\alpha$, $b = c\sinh\alpha$ and $c = \sqrt{(a^2-b^2)}$.

Case I. Let a thin semi-elliptic plate be bounded by the ellipse $\xi = \alpha$ and the major axis. We consider the half which is given by $0 \leqq \xi \leqq \alpha$; $0 \leqq \eta \leqq \pi$. Suppose that a load of weight W is concentrated at the point (γ,δ) where $0 < \gamma < \alpha$ and $0 < \delta < \pi$, and that the elliptic boundary $\xi = \alpha$ is clamped and the straight boundary is merely supported.

The deflexion of the central plane of a thin complete elliptic plate bounded by $\xi = \alpha$, and clamped there, under the action of a load of weight W concentrated at the point γ,δ $(0 < \gamma < \alpha\ ;\ 0 < \delta < \pi)$ is given by (Sengupta, 1949)

$$\omega_1 = -\frac{W}{8\pi D} r_1^2(\log r_1 - \tfrac{1}{2}) + \omega'_1 \tag{1}$$

where $r_1^2 = (x-x_1)^2+(y-y_1)^2$ and (x,y) are the co-ordinates of any point on the plate and (x_1,y_1) are the cartesian co-ordinates of the loaded point whose elliptic co-ordinates are (γ,δ) *i.e.*,

$$x_1 = c\cosh\gamma\cos\delta$$
$$y_1 = c\sinh\gamma\sin\delta$$

and (Timpe, 1923)

$$\begin{aligned}\omega_1' = {} & A_0'+A_0\cosh 2\xi+(A_1'\cosh\xi+A_1\cosh 3\xi)\cos\eta+(B_1'\sinh\xi+B_1\sinh 3\xi)\sin\eta \\ & +(A_0+A_2'\cosh 2\xi+A_2\cosh 4\xi)\cos 2\eta+(B_2'\sinh 2\xi+B_2\sinh 4\xi)\sin 2\eta \\ & +\sum_{n=3}^{\infty}\left[\{A_{n-2}\cosh(n-2)\xi+A_n'\cosh n\xi+A_n\cosh(n+2)\xi\}\cos n\eta\right. \\ & \left.+\{B_{n-2}\sinh(n-2)\xi+B_n'\sinh n\xi+B_n\sinh(n+2)\xi\}\sin n\eta\right]\end{aligned} \tag{2}$$

where

$$2A_0'\sinh 2\alpha = \frac{Wc^2}{16\pi D}\,[(\cosh 2\alpha+\cosh 2\gamma+\cosh 2\delta)(2\alpha\sinh 2\alpha-\cosh 2\alpha)$$

$$-\alpha\sinh 4\alpha - L_0^{(1)}+L_0^{(2)}\sinh 2\alpha+L_0^{(3)}] \tag{3}$$

$$2A_0\sinh 2\alpha = \frac{Wc^2}{16\pi D}\,[(\cosh 2\alpha+\cosh 2\gamma+\cos 2\delta)+2\alpha\sinh 2\alpha$$

$$+L_0^{(1)}\exp.(2\alpha)-L_0^{(3)}\exp.(-2\alpha)] \tag{4}$$

$$A_1'(\sinh 4\alpha+2\sinh 2\alpha) = \frac{Wc^2}{32\pi D}\,[8(\cosh 3\alpha-3\alpha\sinh 3\alpha)\cosh\alpha\cosh\gamma\cosh\delta$$

$$+8\alpha\sinh\alpha\cosh 3\alpha\cosh\gamma\cosh\delta+L_1^{(1)}(3\sinh 3\alpha-\cosh 3\alpha)\exp(\alpha)$$

$$+L_1^{(2)}(3\sinh 3\alpha+\cosh 3\alpha)\exp(-\alpha)+3L_1^{(3)}] \tag{5}$$

$$A_1(\sinh 4\alpha+2\sinh 2\alpha) = -\frac{Wc^2}{32\pi D}[8(\cosh\alpha-\alpha\sinh\alpha)\cosh\alpha\cosh\gamma\cos\delta$$

$$+4\alpha\sinh 2\alpha\cosh\gamma\cos\delta-L_1^{(1)}+L_1^{(2)}+L_1^{(3)}(3\cosh\alpha+\sinh\alpha)\exp.(-3\alpha)] \tag{6}$$

$$B_1'(\sinh 4\alpha-2\sinh 2\alpha) = \frac{Wc^2}{32\pi D}\,[-8(3\alpha\cosh 3\alpha-\sinh 3\alpha)\sinh\alpha\sinh\gamma\sinh\delta$$

$$+8\alpha\sinh 3\alpha\cosh\alpha\sinh\gamma\sin\delta+M_1^{(1)}(3\cosh 3\alpha-\sinh 3\alpha)\exp(\alpha)$$

$$+M_1^{(2)}(3\cosh 3\alpha+\sinh 3\alpha)\exp(-\alpha)+3M_1^{(3)}] \tag{6a}$$

$$B_1(\sinh 4\alpha-2\sinh 2\alpha) = \frac{Wc^2}{32\pi D}\,[8(\alpha\cosh\alpha-\sinh\alpha)\sinh\alpha\sinh\gamma\sin\delta$$

$$-8\alpha\sinh\alpha\cosh\alpha\sinh\gamma\sin\delta-M_1^{(1)}-M_1^{(2)}-M_1^{(3)}(\cosh\alpha+3\sinh\alpha)\exp(-3\alpha)] \tag{7}$$

$$A_2'(\sinh 6\alpha+3\sinh 2\alpha) = -4A_0\sinh 4\alpha+\frac{Wc^2}{16\pi D}\,[(4\alpha\sinh 4\alpha-\cosh 4\alpha)$$

$$+2L_2^{(1)}\sinh 4\alpha+L_2^{(2)}(2\sinh 4\alpha+\cosh 4\alpha)\exp.(-2\alpha)+2L_2^{(3)}] \tag{8}$$

$$A_2(\sinh 6\alpha+3\sinh 2\alpha) = 2A_0\sinh 2\alpha-\frac{Wc^2}{16\pi D}\,[(2\alpha\sinh 2\alpha-\cosh 2\alpha)$$

$$+L_2^{(1)}\sinh 2\alpha+L_2^{(2)}+L_2^{(3)}(\sinh 2\alpha+2\cosh 2\alpha)\exp(-4\alpha)] \tag{9}$$

$$B_2'(\sinh 6\alpha-3\sinh 2\alpha)$$

$$=\frac{Wc^2}{16\pi D}[2M_2^{(1)}\cosh 4\alpha+M_2^{(2)}(2\cosh 4\alpha+\sinh 4\alpha)\exp(-2\alpha)+2M_2^{(3)}] \tag{10}$$

$$B_2(\sinh 6\alpha-3\sinh 2\alpha)$$

$$=-\frac{Wc^2}{16\pi D}[M_2^{(1)}\cosh 2\alpha+M_2^{(2)}+M_2^{(3)}(\cosh 2\alpha+2\sinh 2\alpha)\exp(-4\alpha)] \tag{11}$$

The author regrets that some errors of calculation and misprints appeared in the values of the constants A's and B's contained in the paper (Sengupta, 1949). Correct expression for these appear above. There are certain other obvious misprints which a reader will easily detect.

$$A_n'[\sinh(2n+2)\alpha+(n+1)\sinh 2\alpha] = -A_{n-2}[2\sinh 2n\alpha+n\sinh 4\alpha]$$
$$+\frac{Wc^2}{32\pi D}[L_n^{(1)}\{(n+2)\sinh(n+2)\alpha+(n-2)\cosh(n+2)\alpha\}\exp\{-(n-2)\alpha\}$$
$$+L_n^{(2)}\{(n+2)\sinh(n+2)\alpha+n\cosh(n+2)\alpha\}\exp(-n\alpha)+(n+2)L_n^{(3)}] \quad (12)$$

$$A_n[\sinh(2n+2)\alpha+(n+1)\sinh 2\alpha] = A_{n-2}[\sinh(2n-2)\alpha+(n-1)\sinh 2\alpha]$$
$$-\frac{Wc^2}{32\pi D}[L_n^{(1)}\{n\sinh n\alpha+(n-2)\cosh n\alpha\}\exp\{-(n-2)\alpha\}+nL_n^{(2)}$$
$$+L_n^{(3)}\{n\sinh n\alpha+(n+2)\cosh n\alpha\}\exp\{-(n+2)\alpha\}] \quad (13)$$

$$B_n'[\sinh(2n+2)\alpha-(n+1)\sinh 2\alpha] = -B_{n-2}[2\sinh 2n\alpha-n\sinh 4\alpha]$$
$$+\frac{Wc^2}{32\pi D}[M_n^{(1)}\{(n+2)\cosh(n+2)\alpha+(n-2)\sinh(n+2)\alpha\}\exp\{-(n-2)\alpha\}$$
$$+M_n^{(2)}\{(n+2)\cosh(n+2)\alpha+n\sinh(n+2)\alpha\}\exp(-n\alpha)+(n+2)M_n^{(3)}] \quad (14)$$

$$B_n[\sinh(2n+2)\alpha-(n+1)\sinh 2\alpha] = B_{n-2}[\sinh(2n-2)\alpha-(n-1)\sinh 2\alpha]$$
$$-\frac{Wc^2}{32\pi D}[M_n^{(1)}\{n\cosh n\alpha+(n-2)\sinh n\alpha\}\exp\{-(n-2)\alpha\}+nM_n^{(2)}$$
$$+M_n^{(3)}\{n\cosh n\alpha+(n+2)\sinh n\alpha\}\exp\{-(n+2)\alpha\} \quad (15)$$

and $L_0^{(1)} = \log\tfrac{1}{2}c-\tfrac{1}{2}$ (16)

$$L_0^{(2)} = (2\log\tfrac{1}{2}c+1)(\cosh 2\gamma+\cos 2\delta) \quad (17)$$
$$L_0^{(3)} = \log\tfrac{1}{2}c+\tfrac{3}{2}+\cosh 2\gamma\cos 2\delta \quad (18)$$
$$L_1^{(1)} = -4(\log\tfrac{1}{2}c)\cosh\gamma\cos\delta \quad (19)$$
$$L_1^{(2)} = -4(1+\log\tfrac{1}{2}c)\cosh\gamma\cos\delta-(\cosh 3\gamma\cos\delta+\cosh\gamma\cos 3\delta) \quad (20)$$
$$L_1^{(3)} = \tfrac{1}{3}\cosh 3\gamma\cos 3\delta-\cosh\gamma\cos\delta \quad (21)$$
$$M_1^{(1)} = -(4\log\tfrac{1}{2}c)\sinh\gamma\sin\delta \quad (22)$$
$$M_1^{(2)} = 4(1+\log\tfrac{1}{2}c)\sinh\gamma\sin\delta-(\sinh 3\gamma\sin\delta+\sinh\gamma\sin 3\delta) \quad (23)$$
$$M_1^{(3)} = \tfrac{1}{3}\sinh 3\gamma\sin 3\delta-\sinh\gamma\sin\delta \quad (24)$$
$$L_2^{(1)} = (1+2\log\tfrac{1}{2}c)+\cosh 2\gamma\cos 2\delta \quad (25)$$
$$L_2^{(2)} = -\tfrac{1}{3}(\cosh 4\gamma\cos 2\delta+\cosh 2\gamma\cos 4\delta)+(\cosh 2\gamma+\cos 2\delta) \quad (26)$$
$$L_2^{(3)} = \tfrac{1}{6}\cosh 4\gamma\cos 4\delta-\tfrac{1}{3}\cosh 2\gamma\cos 2\delta \quad (27)$$
$$M_2^{(1)} = \sinh 2\gamma\sin 2\delta \quad (28)$$
$$M_2^{(2)} = -\tfrac{1}{3}(\sinh 4\gamma\sin 2\delta+\sinh 2\gamma\sin 4\delta) \quad (29)$$
$$M_2^{(3)} = \tfrac{1}{6}\sinh 4\gamma\sin 4\delta-\tfrac{1}{3}\sinh 2\gamma\sin 2\delta \quad (30)$$
$$L_n^{(1)} = \frac{2}{n(n-1)}\cosh n\gamma\cos n\delta-\frac{2}{(n-1)(n-2)}\cosh(n-2)\gamma\cos(n-2)\delta \quad (31)$$
$$L_n^{(2)} = -\frac{2}{n(n+1}\{\cosh(n+2)\gamma\cos n\delta+\cosh n\gamma\cos(n+2)\delta\}$$
$$+\frac{2}{n(n-1)}\{\cosh n\gamma\cos(n-2)\delta+\cosh(n-2)\gamma\cos n\delta\} \quad (32)$$

$$L_n^{(3)} = \frac{2}{(n+1)(n+2)} \cosh (n+2)\gamma \cos (n+2)\delta - \frac{2}{n(n+1)} \cosh n\gamma \cos n\delta \qquad (33)$$

$$M_n^{(1)} = \frac{2}{n(n-1)} \sinh n\gamma \sin n\delta - \frac{2}{(n-1)(n-2)} \sinh (n-2)\gamma \sin (n-2)\delta \qquad (34)$$

$$M_n^{(2)} = -\frac{2}{n(n+1)} \{\sinh (n+2)\gamma \sin n\delta + \sinh n\gamma \sin (n+2)\delta\}$$

$$+\frac{2}{n(n-1)} \{\sinh n\gamma \sin (n-2)\delta + \sinh (n-2)\gamma \sin n\delta\} \qquad (35)$$

$$M_n^{(3)} = \frac{2}{(n+1)(n+2)} \sinh (n+2)\gamma \sin (n+2)\delta - \frac{2}{n(n+1)} \sinh n\gamma \sin n\delta \qquad (36)$$

We now place an equal weight W at the point $(\gamma, -\delta)$. The deflexion which we denote by ω_2 may be shown to be given by

$$\omega_2 = -\frac{W}{8\pi D} r_2^2(\log r_2 - \tfrac{1}{2}) + \omega_2'$$

where

$$r_2^2 = (x-x_1)^2 + (y+y_1)^2 = (x - c \cosh \gamma \cos \delta)^2 + (y + c \sinh \gamma \sin \delta)^2$$

and

$$\omega_2' = A_0' + A_0 \cosh 2\xi + (A_1' \cosh \xi + A_1 \cosh 3\xi) \cos \eta - (B_1' \sinh \xi + B_1 \sinh 3\xi) \sin \eta$$

$$+ (A_0 + A_2' \cosh 2\xi + A_2 \cosh 4\xi) \cos 2\eta - (B_2' \sinh 2\xi + B_2 \sinh 4\xi) \sin 2\eta$$

$$+ \sum_{n=3}^{\infty} [\{A_{n-2} \cosh (n-2)\xi + A_n' \cosh n\xi + A_n \cosh (n+2)\xi\} \cos n\eta$$

$$-\{B_{n-2} \sinh (n-2)\xi + B_n' \sinh n\xi + B_n \sinh (n+2)\xi\} \sin n\eta] \qquad (38)$$

the constants A_n, B_n etc. being given by the same formulae as before.

Now, we consider the function

$$\omega = \omega_1 - \omega_2 = f(x, y) + 2\phi(\xi, \eta) \qquad (39)$$

where

$$f(x, y) = -\frac{W}{16\pi D} [\{(x-x_1)^2 + (y-y_1)^2\} \log \{(x-x_1)^2 + (y-y_1)^2\}$$

$$- \{(x-x_1)^2 + (y+y_1)^2\} \log \{(x-x_1)^2 + (y+y_1)^2\} + 4yy_1] \qquad (40)$$

and

$$\phi(\xi, \eta) = (B_1' \sinh \xi + B_1 \sinh 3\xi) \sin \eta + (B_2' \sinh 2\xi + B_2 \sinh 4\xi) \sin 2\eta$$

$$+ \sum_{n=3}^{\infty} \{B_{n-2} \sinh (n-2)\xi + B_n' \sinh n\xi + B_n \sinh (n+2)\xi\} \sin n\eta$$

$$= y \Big[\frac{B_1}{c} + \frac{2B_1}{c^3} (2x^2 + 2y^2 + c^2) + \frac{4B'_2}{c^2} x + \frac{16B_2}{c^4} (x^3 + xy^2)$$

$$+ \frac{B_3'}{c^3} (12x^2 - 4y^2 - 3c^2) + \frac{2B_3}{c^5} \{(24x^4 - 8c^2x^2 - c^4) + 8y^2(2x^2 - c^2) - 8y^4\} + \cdots \Big]^* \qquad (41)$$

---

* The desirability of expressing biharmonic functions $\cosh n\xi \cos (n-2)\eta + \cosh (n-2)\xi \cos n\xi$ in cartesian co-ordinates was suggested by Prof. N. M. Basu. The plane harmonic function $\cosh 3\xi \cos 3\eta$ may be found expressed in cartesian co-ordinates in Love's Mathematical Theory of Elasticity on page 341 (Third edition).

where the expression within the third brackets is a series of polynomials in x and y involving only even powers of y. The series is absolutely and uniformly convergent in a domain extending beyond the elliptic plate and the derivatives of all orders of the function defined by $\phi(\xi, \eta)$ with respect to x and y may be had by term by term differentiation.

Evidently $\nabla_1^4 \omega = 0$, throughout the semi-elliptic plate $0 < \xi < \alpha$; $0 < \eta < \pi$, excepting at the point (γ, δ) *i.e.* (x_1, y_1). The resultant normal shearing stress $\int N ds$ across a sufficiently small closed curve with the point (γ, δ) in its interior clearly balances a load of weight W. Also the conditions $\omega = \partial\omega/\partial\nu = 0$ are satisfied over $\xi = \alpha$. It is easily verified that $f = \phi = \partial^2 f/\partial y^2 = \partial^2\phi/\partial y^2 = 0$ over $y = 0$.

Now, the conditions at a supported edge are $\omega = G = 0$. These at the straight edge $y = 0$, reduce to $\omega = \partial^2\omega/\partial y^2 = 0$. These are evidently satisfied by the function ω as defined by formulae $(39)_4$ over $y = 0$. It, therefore, defines the deflexion of the central plane of a thin semi-elliptic plate $0 \leqq \xi \leqq \alpha$; $0 \leqq \eta \leqq \pi$, made of isotropic elastic material, clamped at the elliptic boundary and merely supported along the major axis under the action of a load of weight W concentrated at any point (γ, δ) $0 < \gamma < \alpha$, $0 < \delta < \pi$, on the upper face of the plate.

Explicit formulae for the constants A_n's and B_n's may be obtained by following the method indicated by Happel (1921). They are

$$
\begin{aligned}
A_{2m}[\sinh(4m+2)\alpha + (2m+1)\sinh 2\alpha] &= A_2(\sinh 6\alpha + 3\sinh 2\alpha) \\
&- \frac{Wc^2}{32\pi D}[\{3L_4^{(1)} + 5L_6^{(1)} + 7L_8^{(1)} + \cdots + (2m-1)L_{2m}^{(1)}\} \exp(2\alpha) \\
&- \{L_4^{(1)} \exp(-6\alpha) + L_6^{(1)} \exp(-10\alpha) + L_8^{(1)} \exp(-14\alpha) + \cdots + L_{2m}^{(1)} \exp\{-(4m-2)\alpha\}\} \\
&+ \{4L_4^{(2)} + 6L_6^{(2)} + 8L_8^{(2)} + \cdots + 2mL_{2m}^{(2)}\} \\
&+ \{5L_4^{(3)} + 7L_6^{(3)} + 9L_8^{(3)} + \cdots + (2m+1)L_{2m}^{(3)}\} \exp(-2\alpha) \\
&+ \{L_4^{(3)} \exp(-10\alpha) + L_6^{(3)} \exp(-14\alpha) + L_8^{(3)} \exp(-18\alpha) \\
&+ \cdots + L_{2m}^{(3)} \exp\{-(4m+2)\alpha\}\}]^*. \qquad (42) \\
&= A_4(\sinh 6\alpha + 3\sinh 2\alpha) \\
&- \frac{Wc^2}{16\pi D}\bigg[\left(\frac{1}{2m}\cosh 2m\gamma \cos 2m\delta - \tfrac{1}{2}\cosh 2\gamma \cos 2\delta\right)\exp(2\alpha) \\
&+ \{\tfrac{1}{3}(\cosh 4\gamma \cos 2\delta + \cosh 2\gamma \cos 4\delta) \\
&- \frac{1}{2m+1}(\cosh(2m+2)\gamma \cos 2m\delta + \cosh 2m\gamma \cos(2m+2)\delta)\}
\end{aligned}
$$

* Using a suggestion kindly offered by Prof. Konrad Ludwing of Hannover (through Prof. N. M. Basu) in another connection (Sengupta, 1948), the above expression is shortened into the form given by formula (48). Prof. Ludwig kindly points out two misprints and an inaccuracy in the same paper. On page 23, line 27, $\cos(n-2)\beta/(n-1)(n-2)$ should be replaced by $-\cos(n-2)\beta/(n-1)(n-2)$; on page 24, line 24, $\exp(-12\alpha)$ by $\exp(-14\alpha)$ and on page 32, line 2, $-Wc^2/32\pi D$ by $-W/8\pi D$. There are still a few more obvious misprints in the above paper, which a reader will easily detect.

$$+\left\{\frac{1}{2m+2}\cosh(2m+2)\gamma\cos(2m+2)\delta-\tfrac{1}{4}\cosh 4\gamma\cos 4\delta\right\}\exp(-2\alpha)$$

$$-\left(\frac{1}{3.4}\cosh 4\gamma\cos 4\delta-\frac{1}{2.3}\cosh 2\gamma\cos 2\delta\right)\exp(-6\alpha)$$

$$+\left\{\frac{1}{(2m+1)(2m+2)}\cosh(2m+2)\gamma\cos(2m+2)\delta\right.$$

$$\left.-\frac{1}{2m(2m+1)}\cosh 2m\gamma\cos 2m\delta\right\}\exp\{-(4m+2)\alpha\}\Bigg]. \tag{43}$$

Similarly

$$A_{2m+1}[\sinh(4m+4)\alpha+(2m+2)\sinh 2\alpha] = A_1(\sinh 4\alpha+2\sinh 2\alpha)$$

$$-\frac{Wc^2}{16\pi D}\Bigg[\left\{\frac{1}{2m+1}\cosh(2m+1)\gamma\cos(2m+1)\delta-\cosh\gamma\cos\delta\right\}\exp(2\alpha)$$

$$+\{\tfrac{1}{2}(\cosh 3\gamma\cos\delta+\cosh\gamma\cos 3\delta)$$

$$-\frac{1}{2m+2}\{\cosh(2m+3)\gamma\cos(2m+1)\delta+\cosh(2m+1)\gamma\cos(2m+3)\delta\}$$

$$+\left\{\frac{1}{2m+3}\cosh(2m+3)\gamma\cos(2m+3)\delta-\tfrac{1}{3}\cosh 3\gamma\cos 3\delta\right\}\exp(-2\alpha)$$

$$-\left(\frac{1}{2.3}\cosh 3\gamma\cos 3\delta-\frac{1}{1.2}\cos\gamma\cos\delta\right)\exp(-4\alpha)$$

$$+\left\{\frac{1}{(2m+2)(2m+3)}\cosh(2m+3)\gamma\cos(2m+3)\delta\right.$$

$$\left.-\frac{1}{(2m+1)(2m+2)}\cosh(2m+1)\gamma\cos(2m+1)\delta\right\}\exp\{-(4m+4)\alpha\}\Bigg] \tag{44}$$

And

$$B_{2m}[\sinh(4m+2)\alpha-(2m+1)\sinh 2\alpha] = B_2[\sinh 6\alpha-3\sinh 2\alpha]$$

$$-\frac{Wc^2}{16\pi D}\Bigg[\left(\frac{1}{2m}\cosh 2m\gamma\cos 2m\delta-\tfrac{1}{2}\sinh 2\gamma\sin 2\delta\right)\exp(2\alpha)$$

$$+\tfrac{1}{3}(\sinh 4\gamma\sin 2\delta+\sinh 2\gamma\sin 4\delta)$$

$$-\frac{1}{2m+1}\{\sinh(2m+2)\gamma\sin 2m\delta+\sinh 2m\gamma\sin(2m+2)\delta\}$$

$$+\left\{\frac{1}{2m+2}\sinh(2m+2)\gamma\sin(2m+2)\delta-\tfrac{1}{4}\sinh 4\gamma\sin 4\delta\right\}\exp(-2\alpha)$$

$$+\left(\frac{1}{3.4}\sinh 4\gamma\sin 4\delta-\frac{1}{2.3}\sinh 2\gamma\sin 2\delta\right)\exp(-6\alpha)$$

$$-\left\{\frac{1}{(2m+1)(2m+2)}\sinh(2m+2)\gamma\sin(2m+2)\delta\right.$$

$$\left.-\frac{1}{2m(2m+1)}\sinh 2m\gamma\sin 2m\delta\right\}\exp\{-(4m+2)\alpha\}\Big]. \qquad (45)$$

$$B_{2m+1}[\sinh(4m+4)\alpha-(2m+2)\sinh 2\alpha] = B_1(\sinh 4\alpha-2\sinh 2\alpha)$$

$$-\frac{Wc^2}{16\pi D}\Big[\left\{\frac{1}{2m+1}\sinh(2m+1)\gamma\sin(2m+1)\delta-\sinh\gamma\sin\delta\right\}\exp(2\alpha)$$

$$+\tfrac{1}{2}(\sinh 3\gamma\sin\delta+\sinh\gamma\sin 3\delta)$$

$$-\frac{1}{2m+2}\{\sinh(2m+3)\gamma\sin(2m+1)\delta+\sinh(2m+1)\gamma\sin(2m+3)\delta\}$$

$$+\left\{\frac{1}{2m+3}\sinh(2m+3)\gamma\sin(2m+3)\delta-\tfrac{1}{3}\sinh 3\gamma\sin 3\delta\right\}\exp(-2\alpha)$$

$$+\left(\frac{1}{2.3}\sinh 3\gamma\sin 3\delta-\frac{1}{1.2}\sinh\gamma\sin\delta\right)\exp(-4\alpha)$$

$$-\left\{\frac{1}{(2m+2)(2m+3)}\sinh(2m+3)\gamma\sin(2m+3)\delta\right.$$

$$\left.-\frac{1}{(2m+1)(2m+2)}\sinh(2m+1)\gamma\sin(2m+1)\delta\right\}\exp\{-(4m+4)\alpha\}\Big]. \qquad (46)$$

Similar compact expressions for A'_{2m}, A'_{2m+1}, B'_{2m} and B'_{2m+1} may now be easily written down.

Case II. Let the loaded semi-elliptic plate be given by $0 \leq \xi \leq \alpha$; $-\frac{1}{2}\pi \leq \eta \leq \frac{1}{2}\pi$, and let the load be placed at the point (γ, δ) where $0 \leq \gamma < \alpha$; $-\frac{1}{2}\pi < \delta < \frac{1}{2}\pi$. We suppose the plate to be clamped along the curved edge and merely supported along the minor axis.

Let us start with the expression for the deflexion of the mid-plane of a complete elliptic plate bounded by $\xi = \alpha$, carrying a load of weight W concentrated at the point (γ, δ) where $0 \leq \gamma < \alpha$; $-\frac{1}{2}\pi < \delta < \frac{1}{2}\pi$. The deflexion of the mid-plane is given by the formula (1) read with the formulae (2), (3), . . . (36).

The deflexion due to a load of weight W concentrated at the point $(\gamma, \pi-\delta)$ is given by

$$\omega_3 = -\frac{W}{8\pi D}\,[r_3^2\log(r_3-\tfrac{1}{2})]+\omega_3' \qquad (47)$$

where

$$r_3^2 = (x+x_1)^2+(y-y_1)^2 \qquad (x_1>0)$$

and

$$\omega_3' = A_0'+A_0\cosh 2\xi-(A_1'\cosh\xi+A_1\cosh 3\xi)\cos\eta$$

$$+(-)^n\sum_{n=2}^{\infty}\{A_{n-2}\cosh(n-2)\xi+A_n'\cosh n\xi+A_n\cosh(n+2)\xi\}\cos n\eta$$

$$+(B_1' \sinh \xi + B_1 \sinh 3\xi) \sin \eta - (B_2' \sinh 2\xi + B_2 \sinh 4\xi) \sin 2\eta$$
$$+(-)^{n+1} \sum_{n=3}^{\infty} \{B_{n-2} \sinh (n-2)\xi + B_n' \sinh n\xi + B_n \sinh (n+2)\xi\} \sin n\eta. \tag{48}$$

We now consider the function

$$\omega = \omega_1 - \omega_3 = -\frac{W}{16\pi D} [\{(x-x_1)^2 + (y-y_1)^2\} \log \{(x-x_1)^2 + (y-y_1)^2\}$$
$$-\{(x+x_1)^2 + (y-y_1)^2\} \log \{(x+x_1)^2 + (y-y_1)^2\} + 4xx_1] + 2(A_1' \cosh \xi + A_1 \cosh 3\xi) \cos \eta$$
$$+2\sum_{n=1}^{\infty} \{A_{2n-1} \cosh (2n-1)\xi + A_{2n+1}' \cosh (2n+1)\xi + A_{2n+1} \cosh (2n+3)\xi\} \cos (2n+1)\eta$$
$$+2(B_2' \sinh 2\xi + B_2 \sinh 4\xi) \sin 2\eta + 2\sum_{n=2}^{\infty} \{B_{2n-2} \sinh (2n-2)\xi$$
$$+ B_{2n}' \sinh 2n\xi + B_{2n} \sinh (2n+2)\xi\} \sin 2n\eta \tag{49}$$

It is easy to verify that the function defined by (49) represents the deflexion of the central plane of the semi-elliptic plate $0 \leqq \xi \leqq \alpha$; $-\frac{1}{2}\pi \leqq \eta \leqq \frac{1}{2}\pi$ clamped at the curved edge and merely supported along the minor axis under the action of a load of weight W concentrated at any point $(\gamma, \delta)(0 \leqq \gamma < \alpha;\ -\frac{1}{2}\pi < \delta < \frac{1}{2}\pi)$ of the plate.

Case III. The deflexion of the central plane of the semi-elliptic plate $0 \leqq \xi \leqq \alpha$; $0 \leqq \eta \leqq \pi$, clamped along the elliptic boundary and merely supported along the major axis under the action of a load of weight W concentrated at the point (x_1, y_1) where $y_1 > 0$ and $0 < x_1^2/a^2 + y_1^2/b^2 < 1$ or in elliptic co-ordinates at (γ, δ) where $0 < \gamma < \alpha$ and $0 < \delta < \pi$ is given by

$$\omega_4 = f(x, y) + 2\phi(\xi, \eta) \tag{50}$$

where

$$f(x, y) = -\frac{W}{16\pi D} [\{(x-x_1)^2 + (y-y_1)^2\} \log \{(x-x_1)^2 + (y-y_1)^2\}$$
$$-\{(x-x_1)^2 + (y+y_1)^2\} \log \{(x-x_1)^2 + (y+y_1)^2\} + 4yy_1] \tag{51}$$

and

$$\phi(\xi, \eta) = (B_1' \sinh \xi + B_1 \sinh 3\xi) \sin \eta + (B_2' \sinh 2\xi + B_2 \sinh 4\xi) \sin 2\eta$$
$$+\sum_{n=3}^{\infty} \{B_{n-2} \sinh (n-2)\xi + B_n' \sinh n\xi + B_n \sinh (n+2)\xi\} \sin n\eta. \tag{52}$$

Let us suppose that (x_1, y_1) lies in the first quadrant, *i.e.* $x_1 > 0$, $y_1 > 0$ and $0 < x_1^2/a^2 + y_1^2/b^2 < 1$. Referring to elliptic coordinates, then $0 < \gamma < \alpha$ and $0 < \delta < \frac{1}{2}\pi$.

We now suppose that another load of weight W is concentrated at the point $(-x_1, y_1)$ *i.e.* at $(\gamma, \pi-\delta)$. Denoting the corresponding deflexion by ω_5. We have

$$\omega_5 = f_2(x, y) + 2\phi_2(\xi, \eta) \tag{53}$$

where

$$f_2(x, y) = -\frac{W}{16\pi D} [\{(x+x_1)^2 + (y-y_1)^2\} \log \{(x+x_1)^2 + (y-y_1)^2\}$$
$$-\{(x+x_1)^2 + (y+y_1)^2\} \log \{(x+x_1)^2 + (y+y_1)^2\} + 4yy_1] \tag{54}$$

and

$$\phi_2(\xi, \eta) = (B_1' \sinh \xi + B_1 \sinh 3\xi) \sin \eta - (B_2' \sinh 2\xi + B_2 \sinh 4\xi) \sin 2\eta$$
$$+ \sum_{n=3}^{\infty} (-)^{n+1} \{B_{n-2} \sinh (n-2)\xi + B_n' \sinh n\xi + B_n \sinh (n+2)\xi\} \sin n\eta. \quad (55)$$

We now consider the function

$$\omega = \omega_4 - \omega_5 = -\frac{W}{16\pi D} [\{(x-x_1)^2 + (y-y_1)^2\} \log \{(x-x_1)^2 + (y-y_1)^2\}$$
$$+ \{(x+x_1)^2 + (y+y_1)^2\} \log \{(x+x_1)^2 + (y+y_1)^2\} - \{(x-x_1)^2 + (y+y_1)^2\} \log \{(x-x_1)^2 + (y+y_1)^2\}$$
$$- \{(x+x_1)^2 + (y-y_1)^2\} \log \{(x+x_1)^2 + (y-y_1)^2\}] + 4(B_2' \sinh 2\xi + B_2 \sinh 4\xi) \sin 2\eta$$
$$+ 4 \sum_{n=2}^{\infty} \{B_{2n-2} \sinh (2n-2)\xi + B_{2n}' \sinh 2n\xi + B_{2n+2} \sinh (2n+2)\xi\} \sin 2n\eta \quad (56)$$

It is like-wise easy to verify that the function defined by (56) represents the deflexion of the mid-plane of the quadrant of a thin elliptic plate given by $0 \leqq \xi \leqq \alpha$; $0 \leqq \eta \leqq \frac{1}{2}\pi$ clamped along the elliptic boundary and merely supported along the straight edges under the action of a load W concentrated concentrated at any point (γ, δ) $(0 < \gamma < \alpha$; $0 < \delta < \frac{1}{2}\pi$ of the upper face of the plate.

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NOTE ON THE ZEROS OF MODIFIED BESSEL FUNCTION DERIVATIVES

By

A. Charnes, *Pittsburg, Pennsylvania, U. S. A.*

(*Communicated by the Secretary—Received November 15, 1950*)

Zeros of modified Bessel functions and their derivatives are respectively required in solutions of boundary value problems of first and second kind in hydrodynamics and wave propagation. While information is available on zeros of the functions, none appears to be on hand for zeros of derivatives. We adduce such in the following for the $K_\nu'(z)$, the notation being that of Watson's "*Bessel Functions*".

A. Location of Zeros

From the differential equation for $K_\nu(z)$ we obtain, for the real part of $(\lambda+\mu)>0$,

$$(\lambda^2-\mu^2)\int_x^\infty xK_\nu(\lambda x)K_\nu(\mu x)dx = x\{\mu K_\nu(\lambda x)K_\nu'(\mu x)-\lambda K_\nu(\mu x)K_\nu'(\lambda x)\}$$

On setting $\mu=\bar{\lambda}$ (so that $\lambda+\mu=2R(\lambda)$) and $x=1$, and since $K_\nu'(z)$ takes conjugate values for conjugate argument, if λ were a zero of $K_\nu'(z)$ we would have

$$\int_1^\infty x\,|K_\nu(\lambda x)|^2dx = 0$$

which is impossible. Hence, there are no zeros with positive real part.

On the imaginary axis,

$$K_\nu'(re^{i\pi/2})K_\nu'(re^{-i\pi/2}) = (\pi/2)^2\{J_\nu'^2(r)+Y_\nu'^2(r)\}.$$

So there are no zeros on the imaginary axis.

From the expression,

$$K_\nu(Z)K_\nu(z) = (1/2)\int_0^\infty \exp\left[-(v/2)-(Z^2+z^2)/2v\right]K_\nu(Zz/v)dv/v$$

valid for $|\arg(Z+z)|<\pi/4$, $|\arg Z|<\pi$, $|\arg z|<\pi$, we obtain,

$$|K_\nu'(re^{i\theta})|^2 = (1/2)\int_0^\infty \exp(\beta)\{[K_\nu(\alpha)+K_\nu''(\alpha)]\alpha-K_\nu'(\alpha)\gamma\}dv/v,$$

where $\alpha=r^2/v$, $\beta=-v/2-r^2\cos 2\theta/v$, $\gamma=(2r^2\,\mathrm{cas}\,2\theta-v)/v^2$. For $3\pi/4\leq|\theta|\leq\pi$, $\cos 2\rho\geq 0$, and the first and third terms in the curly bracket are greater than zero. Also, we have

$$K_\nu''(\alpha)(\alpha/v)+K_\nu'(\alpha)(1/v) = r^{-2}[a^2+v^2]K_\nu(\alpha)$$

from the differential equation. This is positive for $x>0$, hence the integrand is too and there are no zeros for $3\pi/4\leq|\theta|<\pi$.

From the expression for $K_\nu'(re^{\pm\pi i})$, it follows that there are no zeros for $|\arg z| = \pi$ unless $\nu - 1/2$ is an integer, in which case the $K_\nu'(z)$ are simple elementary functions.

B. Number of Zeros

By applying the principle of the argument in a similar fashion to that of Watson but to the function $z^{\nu+1}K_\nu'(z)$, we obtain analogously that the number of zeros of $K_\nu'(z)$ in $|\arg z| < \pi$ is $\nu + 1/2 + (1/\pi)$ arctan $(\cot(\nu+1)\pi)$ *i.e.* the greatest even integer nearest to $\nu + 1/2$.

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ON SOME GEOMETRICAL CONFIGURATIONS—I

By

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(*Received January 16, 1951*)

1. According to the well-known Erlanger Programm, different geometries are defined in terms of different groups of transformations. The groups of transformations which give rise to the different types of classical geometries are of infinite order. A finite geometrical system is obtained by taking a finite number of geometrical objects, satisfying the usual incidence relations and admitting a group of transformations of finite order as the group of automorphisms (Carmichael, 1937)

Felix Klein (Klein, 1913) investigated the finite groups of symmetries of regular solids and showed that the basic symmetries of these solids are determined by the number of generators of the said group and by the orders of these generators. The object of this paper is to study some geometrical configurations which admit a given group of transformations of finite order, generated by two elements, as the group of automorphisms.

In a paper published recently, R. N. Sen (Sen, 1950) considers a group of finite order generated by two distinct elements (operations) each being of order two. The group is then extended to a system of double composition by the introduction of a second composition, which is commutative but not necessarily associative, such that the left-hand distributive law is satisfied†. This system, however, is not closed with respect to either of the compositions. If the group-composition and the second composition be taken as multiplication and addition respectively, then the existence and uniqueness of a "null element", defined by the property $a.\,0 = 0$, for every a belonging to the group, is possible in some special cases. In the present paper a set of transformations isomorphic to the abstract system stated above will be utilised in constructing some geometrical configurations possessing a centre of symmetry (corresponding to the "null element").

2. Let G be a group generated by two distinct elements h and g, different from 1, and satisfying

$$h^2 = g^2 = 1. \tag{1}$$

By virtue of (1), every element or G can be written in one of the forms.

$$1,\ h,\ g,\ hg,\ gh,\ hgh,\ ghg,\ \ldots,$$

or, $1,\ h,\ k^t,\ hk^t,\ k^th,\ hk^th$, where $k = hg$, $t = 1, 2, \ldots$. Now since $hk^t = 1$, or $k^th = 1$ implies either $g = 1$, or $h = 1$, and $hk^th = 1$ implies $k^t = 1$, it follows that G can be of finite order if and only if $k^r = 1$ for some finite r. In that case every element of G can be put in one and only one of the forms:

† In Sen's paper, the generators of the group have been denoted by $*$, 1 and the second composition of the system by '•'.

$$1,\ h,\ k^t,\ k^t h, \quad t = 1, 2, \ldots, r-1, \tag{2}$$

so that the order of G will be equal to $2r$.

The group G is now extended to a system A of a double composition by the introduction of a second composition (besides the group composition), defined for the elements of G only. This composition is assumed to be commutative but not necessarily associative. It is further assumed that the system A admits the elements of G as left-hand operators. That is to say, if the second composition be written as addition,

$$h(g_1+g_2) = hg_1+hg_2\ ;\ g(g_1+g_2) = gg_1+gg_2, \tag{3}$$

where g_1 and g_2 are any two elements of G.

The elements of A, therefore, are of the forms:

$$1,\ h,\ k^t,\ k^th,\ 1+h,\ 1+k^t,\ 1+k^th,\ h+k^t,\ h+k^th,\ k^s+k^th,\ 1+1,\ h+h,\ k^s+k^t,\ k^sh+k^th. \tag{4}$$

R. N. Sen has shown that in some cases there exists a unique element 0 in A, to be called henceforth the null element of A, such that

$$h.0 = 0,\ g.0 = 0, \tag{5}$$

so that $a.0 = 0$, for every a belonging to G. The problem of finding the necessary and sufficient conditions for the existence of such a null element, which has been left open by R. N. Sen, will now be considered.

Since h and g are different from 1, no element of G can remain unaltered when multiplied by h or g. Hence a null element of A, if it exists, must be of one of the last ten types of elements in (4). Now,

$$\begin{array}{ll}
h(1+h) = h+1\ ; & g(1+h) = g+gh = k^{r-1}h+k^{r-1} \\
h(1+k^t) = h+hk^t = h+k^{r-t}h\ ; & g(1+k^t) = g+gk^t = k^{r-1}h+k^{r-t-1}h \\
h(1+k^th) = h+hk^th = h+k^{r-t}\ ; & g(1+k^th) = g+gk^th = k^{r-1}h+k^{r-t-1} \\
h(h+k^t) = 1+hk^t = 1+k^{r-t}h\ ; & g(h+k^t) = gh+gk^t = k^{r-1}+k^{r-t-1}h \\
h(h+k^th) = 1+hk^th = 1+k^{r-t}\ ; & g(h+k^th) = gh+gk^th = k^{r-1}+k^{r-t-1} \\
h(k^s+k^th) = hk^s+hk^th = k^{r-s}h+k^{r-t}\ ; & g(k^s+k^th) = gk^s+gk^th = k^{r-s-1}h+k^{r-t-1} \\
h(k^s+k^t) = hk^s+hk^t = k^{r-s}h+k^{r-t}h\ ; & g(k^s+k^t) = gk^s+gk^t = k^{r-s-1}h+k^{r-t-1}h \\
h(k^sh+k^th) = hk^sk+hk^th = k^{r-s}+k^{r-t}\ ; & g(k^sh+k^th) = gk^sh+gk^th = k^{r-s-1}+k^{r-t-1} \\
h(1+1) = h+h\ ; & g(1+1) = g+g = k^{r-1}h+k^{r-1}h \\
h(h+h) = 1+1\ ; & g(h+h) = gh+gh = k^{r-1}+k^{r-1}.
\end{array}$$

Thus in order that a unique null element 0, satisfying (5), may exist, it is necessary and sufficient to define the second composition, *viz.*, addition, in such a way that one of the following should hold:

(i) $1+h = k^{r-1}h+k^{r-1}$

(ii) $1+k^t = h+k^{r-t}h = k^{r-1}h+k^{r-t-1}h$

(iii) $1+k^th = h+k^{r-t} = k^{r-1}h+k^{r-t-1}$

(iv) $h+k^t = 1+k^{r-t}h = k^{r-1}+k^{r-t-1}h$

(v) $h+k^th = 1+k^{r-t} = k^{r-1}+k^{r-t-1}$

(vi) $k^s+k^t = k^{r-s}h+k^{r-t}h = k^{r-s-1}h+k^{r-t-1}h$

(xii) $k^s+k^th = k^{r-s}h+k^{r-t} = k^{r-s-1}h+k^{r-t-1}$

(viii) $k^sh+k^th = k^{r-s}+k^{r-t} = h^{r-s-1}+k^{r-t-1}$

(ix) $1+1 = h+h = k^{r+1}h+k^{r-1}h$

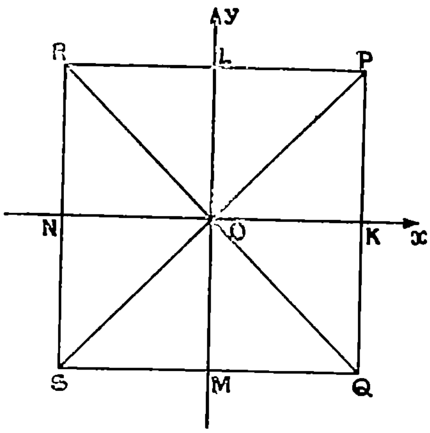
(x) $h+h = 1+1 = k^{r-1}+k^{r-1}$.

The extent to which the algebraic system A is characterised by assuming one or other of these ten conditions will be considered in a following paper. In what follows a geometrical model of the system A, in which one of the above conditions is satisfied, will be given.

8. Let h be the reflection in the x-axis, and g the reflection in the origin ; then both h and g are involutory, and

$$h: \left.\begin{array}{l} x' = x \\ y' = -y \end{array}\right\}, \quad g: \left.\begin{array}{l} x' = -x \\ y' = -y \end{array}\right\}, \quad \text{then } k = hg \cdot \left.\begin{array}{l} x' = -x \\ y'' = y \end{array}\right\}.$$

$k^3 = 1$; hence the group G generated by h and g is of order 4, consisting of the elements 1, h, k, kh. Applying these transformations on a point $P = (x,y)$, the configuration $PQRS$, as shown in the figure is obtained The point P, Q, R, S represent the elements 1, h, k, kh of G. Now, let us define the sum of two points as their mid-point and thus obtain 5 new points K, L, M, N, O, which represent $1+h$, $1+k$, $h+kh$, $k+kh$, $1+kh = h+k$ respectively. Conditions (iii) and (iv) of Art. 2, *viz.*, $1+kh = h+k$, for the existence of a null element is satisfied. The corresponding point 0, which remains invariant by all the transformations of G, is the centre of symmetry of the resulting configuration (which represents the system A of double composition obtained by the adjunction of the sum-elements to the group G). It is interesting to note that if the system A be extended to a system A_1 by adjoining the sum of the elemente of A and similarly A_1 be extended to A_2, and so on, the point 0 remains the centre of symmetry of all the configurations A_i, $i = 1, 2, \ldots$ Finally the system S, consisting of an infinite number of elements, obtained by extending the group G in such a way that S becomes closed with respect to addition, also possesses the very same point 0 as the null element (centre of symmetry).



Configurations having any prescribed value of r and possessing a centre of symmetry are obtained by taking h to be the reflection in the line $y = x \tan\phi$ and g to be the reflection in the x-axis. Starting with the point P at unit distance on the x-axis, the sucessive points obtained for the configuration all lie on the unit circle and correspond

to the angles: $0, 2\phi, -2\phi, 4\phi, -4\phi, 6\phi, -6\phi, \ldots$. For a finite configuration with $2r$ points, ϕ should be taken equal to π/r. If the sum of two points be defined as their mid-point then the origin, which is obtained as a sum-point, is the centre of symmetry of the configuration.

I wish to express my thanks to Dr. R. N. Sen for his suggestion of the problem and encouragement in the preparation of this paper.

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ON THE STRUCTURE OF JOACHIMSTAL'S CIRCLES OF A CONIC

By

K. Rangaswami Aiyer, *Annamalainagar*

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1. With each point P in the plane of a conic Σ may be associated its "pedal tetrad", namely, the feet of the four normals from P to Σ. Three of the four feet determine a "pedal triad" and its circum-circle is called a Joachimstal's circle (Casey p. 218) of Σ. It follows, therefore, that the totality of Joachimstal's circles of the conic constitute an ∞^2-domain in circle space, the groups of four Joachimstal's circles determined by a pedal tetrad having special relations.

Taking the familiar mode (Baker, pp 6, 12, 13) of representing the circles in the plane by the points of a projective three-space, wherein the point-circles are represented by the points on a quadric surface called the "Absolute" we show that the Joachimstal's circles of a conic correspond to the points of a rational quartic scroll (Edge pp. 55-59, Salmon pp. 210-211) in the representative three-space.

2. Let A be any point on the conic Σ, centre O; A', the diametrically opposite point of A and F the foot of the perpendicular from O on the tangent at A' to Σ. If P be any point on the normal at A and B, C, D the feet of the other three normals from P to Σ, it is well known (Casey, *loc. cit.* pp. 217, 219) that as P moves on the normal at A, the Joachimstal's circles BCD generate a coaxal system whose common points are A' and F. We call this system the *associated coaxal system* of A. Now, the Joachimstal's circles of the conic Σ form an ∞^2-system and therefore, correspond to the points of a surface Γ in the representative three space. Since a coaxal system of circles is represented by a line, it follows that the coaxal system of Joachimstal's circles associated with a point, A on Σ corresponds to a generator of Γ.

Let A and A_1 be two points on Σ; it is known that the coaxal system of Joachimstal's circles associated with A and A_1 have no member in common. Further, as there is one coaxal system associated with each point on Σ we see that the corresponding lines in [3] are all generators of Γ; thus Γ is a ruled surface. We now proceed to show that the surface Γ in [3] is a rational quartic scroll.

3. Now, a ruled quartic surface, in general, is generated by establishing a $(2-2)$ correspondence between two skew lines α and β, so that to every point P on α there corresponds two points Q_1 and Q_2 on β and *vice versa*; the lines joining P to Q_1 and Q_2 are generators of the surface (Edge, *loc. cit.* p. 58, 66). Thus, through each point on either of the lines α and β there pass two generators meeting the other line. The lines α and β are the directrices of the surface. In such a general $(2-2)$ correspondence there will be four points of either line giving rise to pairs of coincident points of the other line. If, however, the correspondence is specialised so as to give double generator, the ruled surface

is rational (Edge, *loc. cit.* pp. 59, 64, Salmon, *loc. cit.* p 211) since, then, every plane through the double generator meets the suface in a conic section. Thus, if one of the planes through the double generator meets the surface again in a conic γ, the quartic scroll is generated by the lines meeting the directrices α and β and also the conic γ; the double generator being the line joining the points of intersection of α and β with the plane of γ.

To determine the two coaxal systems corresponding to the directrices α and β of the surface Γ we notice that the line of centres of the coaxal system of Joachimstal's circles associated with A is the perpendicular bisector of $A'F$ since these circles pass through A and F (Casey, *loc. cit.*, p. 219). If this perpendicular bisector meets the major axis of the conic in X and the minor axis in Y and A_1 and A_2 be the reflections of A in the major and minor axis respectively, it is clear by symmetry that the coaxal systems associated with A and A_1 have in common the circle centre X and passing through A' and F while the coaxal systems associated with A and A_2 have in common the circle centre Y and passing through A' and F. Further, it is easily seen that the circles whose centres are X and Y, respectively pass through the extremities of the minor and major axes. In symbols. if A' be the point $(a\cos\theta, b\sin\theta)$ on the ellipse given by the canonical equation $b^2x^2+a^2y^2 = a^2b^2$, the Joachimstal's circles associated with A and whose centres are respectively X and Y are given by the equations:

$$x^2+y^2-\frac{(a^2-b^2)}{a}\cos\theta x - b^2 = 0 \tag{I}$$

and

$$x^2+y^2+\frac{(a^2-b^2)}{b}\sin\theta y - a^2 = 0. \tag{II}$$

Thus, from equations (I) and (II) it follows that the coaxal system of circles whose common points are the extremities of the minor and major axes of the conic correspond to the directrices α and β of Γ. Furthermore, from what is said above it is clear that if X be any point on the major axis and (X) the circle of the system α having X for centre the two coaxal systems of Joachimstal's circles containing (X) will be the associated coaxal systems of A and A_1, besides, each of these coaxal systems will have a member belonging to β say (Y) and (Y_1). Similarly given any circle (Y) of the system β, with centre Y on the minor axis, the two coaxal systems of circles associated with A and A_2 will contain (Y) and each system will have a member in common with the system α. Thus the correspondence between the circles (X) and (Y) on α and β is seen to be a $(2-2)$ correspondence. We now proceed to show that the lines joining corresponding points in this correspondence meet also a conic section γ.

To see this we notice that the radical axis of the coaxal system of Joachimstal's circles associated with A on Σ is the tangent at A' to Σ. Now, as there is only one line circle in a general coaxal system, it is obvious that the tangents to Σ are the Joachenistal's line circles and their representaive points in [3] lie on Γ. These line circles are orthogonal to the circle point at infinity in [3] and besides, as only two tangents to Σ pass through any point, there are just two Joachimstal's line circles

orthogonal to any circle. Thus in the representative three space the Joachimstal's line circles are represented by the points of a conic γ which lies on Γ. Further more, it is clear that the coaxal system of Joachimstal's circles associated with any point A on Σ, which corresponds to a generator of Γ meets the conic γ in just one point and cuts the directries α and β in the points corresponding to the circles X and Y. As the coaxal systems of circles α and β have no member in common, the lines α and β are skew and meet the plane of γ in two points U and V which correspond respectively to the minor and major axes of the conic Σ while the two points in which the line UV meets γ correspond to the asymptotes of Σ. Thus Γ is a quartic scroll having α and β for directrices and UV for a double generator.

4. Now any plane π through the double generator UV cuts Γ in a conic, the points of which correspond to the Joachimstal's circles orthogonal to some circle concentric with Σ; that is orthogonal to the director circle of some conic Σ_1 coaxal with Σ. If BCD be a triangle inscribed in Σ and self polar for Σ_1, then BCD is apolar to the contact tetrad of Σ and Σ_1 which form the vertices of an inscribed rectangle of Σ (Vaidyanathswamy pp. 296-302). Thus BCD is a pedal triad and moreover its circumcircle, which is a Joachimstal's circle is by Gaskin's theorem, orthogonal to the director circle of Σ_1. Hence the points of the conic in which the plane π meets Γ correspond to the circum-circles of triangles inscribed in Σ and self polar for a conic Σ_1 coaxal with Σ (Vaidyanathswamy, *loc. cit.* p. 301).

We now proceed to derive the equation to the surface Γ in the representative three space, by taking for homogeneous coordinates of the space the special tetracyclic coordinates* referred to the four circles, namely, the circles on the major and minor axes as diameters and the principal axes of the conic Σ regarded as line circles. Now, the coaxal system of Joachimstal's circles associated with any point A on Σ corresponds to a generator of Γ and in this coaxal system there are two circles (X) and (Y) belonging to the system α and β respectively. Thus, the equation to any circle of the system associated with A can, by equations (I) and (II) be written in the form:

$$\rho\left\{x^2+y^2-\frac{(a^2-b^2)}{a}x\cos\theta-b^2\right\}+\rho'\left\{x^2+y^2+\frac{a^2-b^2}{b}y\sin\theta-a^2\right\}=0 \qquad \text{(III)}$$

which may be rewritten in the form:

$$\rho\{x^2+y^2-b^2\}+\rho'\{x^2+y^2-a^2\}-\rho\frac{a^2-b^2}{a}x\cos\theta+\rho'\frac{a^2-b^2}{b}y\sin\theta=0 \qquad \text{(IV)}$$

so that the tetracyclic coordinates of the circle (III) are:

$$\xi=\rho';\ \eta=\rho,\ \zeta=-\rho\frac{a^2-b^2}{a}\cos\theta,\ \tau=\rho'\frac{a^2-b^2}{b}\sin\theta. \qquad \text{(V)}$$

Eliminating θ we have the equation to the surface Γ in the form:

* If $C_1\equiv0$; $C_2\equiv0$; $C_3\equiv0$; $C_4\equiv0$ be the canonical equations of four circles, then the special tetracyclic coordinates of any circle $\xi C_1+\eta C_2+\zeta C_3+\tau C_4\equiv0$ are (ξ,η,ζ,τ) referred to these four fundamental circles.

$$a^2\xi^2\zeta^2 + b^2\eta^2\tau^2 = (a^2 - b^2)^2\xi^2\eta^2. \tag{VI}$$

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TRANSVERSE VIBRATION OF A WOODEN PLATE

By

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Timoshenko (1922) gave an exact solution of transverse vibration of a thin strip of isotropic plate. But wooden plates cannot be considered as isotropic.

If from an orthotropic material a flat plate is cut parallel to a plane of elastic symmetry, it will have two perpendicular axes of symmetry in the plane of the plate. An example of this kind of plate is a plain-sawn board. The x axis and y-axis are taken parallel to axes of symmetry. The problem will be treated as one of plane stress.

We can assume the components of stress and strain in an orthotropic plate as connected by the following relations (*cf.* Bassel Smith, 1949):

$$\left.\begin{aligned} e_{xx} &= \frac{1}{E_x} X_x - \frac{\sigma_{yx}}{E_y} Y_y \\ e_{yy} &= \frac{1}{E_y} Y_y - \frac{\sigma_{xy}}{E_x} X_x \\ e_{xy} &= \frac{1}{\mu_{xy}} X_y \end{aligned}\right\} . \tag{1}$$

In these equations E_x and E_y are Young's moduli in the $x-$ and $y-$ directions respectively σ_{xy} is the ratio of contraction parallel to y-axis to the extension parallel to x-axis and similarly σ_{yx} is the ratio of contraction parallel to x-axis to the extension parallel to y-axis. The quantity μ_{xy} is the modulus of rigidity associated with $x-$ and $y-$ directions. In the case of plane-sawn plate of Stikka-spruce the experimental results are as follows (*cf.* B. Smith, 1949):

$$\left.\begin{aligned} E_x &= 1{\cdot}679 \times 10^6 \quad \text{lb}/in^2 \\ E_y &= {\cdot}076 \times 10^6 \quad \text{lb}/in^2 \\ \mu_{xy} &= {\cdot}112 \times 10^6 \quad \text{lb}/in^2 \\ \sigma_{xy} &= {\cdot}464 \end{aligned}\right\} \tag{2}$$

For the type of plates we have considered $\sigma_{xy}/E_x = \sigma_{yx}/E_y$. Henceforward E_x, E_y, σ_{xy}, σ_{yx} will be denoted by E_1, E_2, σ_2, σ_1 respectively.

The equations of motion are,

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} \\ \rho\frac{\partial^2 v}{\partial t^2} &= \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} \end{aligned}\right\} \tag{3}$$

Let

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \chi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} - \frac{\partial \chi}{\partial x} \tag{4}$$

and each of u, v contain a factor $\cos(pt)$. So from (3) and (4),

$$\left.\begin{aligned} -\rho p^2(\phi_x + \chi_y) &= A\{\phi_{xxx} + (1-\sigma_1)\chi_{xxy} + \sigma_1\phi_{xxy}\} + C\{2\phi_{xxy} + \chi_{yyy} - \chi_{xxy}\} \\ -\rho p^2(\phi_y - \chi_x) &= C\{2\phi_{xxy} + \chi_{xyy} - \chi_{xxx}\} + B\{\phi_{yyy} - \chi_{xyy} + \sigma_2\phi_{xxy} + \sigma_2\chi_{xyy}\} \end{aligned}\right\} \tag{5}$$

where, $A = \dfrac{E_1}{1-\sigma_1\sigma_2}$, $B = \dfrac{E_2}{1-\sigma_1\sigma_2}$ and $C = \mu$ and the suffixes denote partial differentiations with respect to that variable.

It is assumed,

$$\left.\begin{aligned} \phi &= R\sin(\alpha x)\sinh(\alpha q y) \\ \chi &= S\cos(\alpha x)\cosh(\alpha q y) \end{aligned}\right\} \tag{6}$$

where R, S, q, α are constants. On substitution of (6) in (5),

$$\left.\begin{aligned} R\{(A-\rho n^2) - (2C + A\sigma_1)q^2\} + qS\{(A-\rho n^2) - (A\sigma_1 + C) - Cq^2\} &= 0 \\ qR\{(\rho n^2 - 2C - B\sigma_2) + Bq^2\} - S\{(C-\rho n^2) + q^2(C - B + B\sigma_2)\} &= 0 \end{aligned}\right\} \tag{7}$$

where $n^2 = p^2/\alpha^2$.

Eliminating R, S from (7),

$$\begin{aligned} &\{(A-\rho n^2) - (2C + A\sigma_1)q^2\}\{(C-\rho n^2) + q^2(C - B + B\sigma_2)\} \\ &\quad + q^2\{(A-\rho n^2) - (A\sigma_1 + C) - Cq^2\}\{(\rho n^2 - 2C - B\sigma_2) + Bq^2\} = 0. \end{aligned} \tag{8}$$

Let

$$\left.\begin{aligned} A - \rho n^2 &= G \\ C - \rho n^2 &= H \\ C + A\sigma_1 &= L \\ C + B\sigma_2 &= M \end{aligned}\right\}. \tag{9}$$

Equation (8) is therefore,

$$(G - Lq^2 - Cq^2)(H + Mq^2 - Bq^2) + q^2(G - L - Cq^2)(-H - M - Bq^2) = 0$$

or,

$$(1-q^2)\{CBq^4 - q^2(GB - LM + CH) + GH\} = 0 \tag{10}$$

It will be shewn later that the solution $q^2 = 1$ of equation (10) does not give a relevant solution of the problem. Therefore,

$$CBq^4 - q^2(GB - LM + CH) + GH = 0. \tag{11}$$

Let q_1 and q_2 be two positive real roots of equation (11), it being assumed $\rho n^2 < C < A$. In fact it can be shewn for Stikka-spruce the equation (11) has two positive real roots. For in that case,

$$\begin{aligned} A &= 1{\cdot}715 \times 10^6 \text{ lbs}/in^2 \\ B &= {\cdot}077 \times 10^6 \text{ lbs}/in^2 \\ C &= {\cdot}112 \times 10^6 \text{ lbs}/in^2 \\ \sigma_{xy} &= {\cdot}464 \\ \sigma_{yx} &= {\cdot}021, \quad \text{so that} \end{aligned}$$

$GB-LM+CH = \cdot 124\times 10^6 - \cdot 189\rho n^2$ which is positive since $\rho n^2 < C$ i.e. 112×10^6
Therefore, from (6),

$$\left.\begin{aligned} \phi &= \sin(\alpha x)\{R_1 \sinh(\alpha q_1 y)+R_2 \sinh(\alpha q_2 y)\} \\ \chi &= \cos(\alpha x)\{S_1 \sinh(\alpha q_1 y)+S_2 \sinh(\alpha q_2 y)\} \end{aligned}\right\}, \qquad (12)$$

R_1, S_1, R_2, S_2 being constants. The boundary conditions are that the surfaces $y=\pm c$ must be free from traction i.e. $X_y = Y_y = 0$ at $y=\pm c$. Then,

$$\left.\begin{aligned} \phi_{yy}+\sigma_2\phi_{xx}-(1-\sigma_2)\chi_{xy} &= 0 \\ 2\phi_{xy}+\chi_{yy}-\chi_{xx} &= 0 \end{aligned}\right\} \qquad (13)$$

at $y=\pm c$ and for all x. From (7),

$$R_1 = -S_1 m_1^{-1} q_1 \quad \text{and} \quad R_2 = -S_2 m_2^{-1} q_2$$

where,

$$m_1^{-1} = -\frac{(A-\rho n^2)-(A\sigma_1+C)-Cq_1^2}{(A-\rho n^2)-(2C+A\sigma_1)q_1^2} \qquad (14)$$

and similar expression for m_2^{-1}, q_2 being substituted for q_1. Therefore from (12) and (13),

$$\left.\begin{aligned} R_1 \sinh(\alpha q_1 c)\{q_1^2-\sigma_2+(1-\sigma_2)m_1\}+R_2\sinh(\alpha q_2 c)\{q_2^2-\sigma_2+(1-\sigma_2)m_2\} &= 0 \\ R_1\cosh(\alpha q_1 c)\{2q_1+m_1 q_1+m_1 q_1^{-1}\}+R_2\cosh(\alpha q_2 c)\{2q_2+m_2 q_2+m_2 q_2^{-1}\} &= 0 \end{aligned}\right\} \qquad (15)$$

The frequency equation is obtained by eliminating R_1, R_2 from equation (15) i.e.

$$\begin{aligned} &\{q_1^2-\sigma_2+(1-\sigma_2)m_1\}\{2q_2+m_2 q_2+m_2 q_2^{-1}\}\tanh(\alpha q_1 c) \\ &\qquad = \{q_2^2-\sigma_2+(1-\sigma_2)m_2\}\{2q_1+m_1 q_1+m_1 q_1^{-1}\}\tanh(\alpha q_2 c). \end{aligned} \qquad (16)$$

For an isotropic material,

$$\left.\begin{aligned} A &= B = \lambda+2\mu \\ C &= \mu \\ \sigma_1 &= \sigma_2 = \frac{\lambda}{\lambda+2\mu} \\ G &= \lambda+2\mu-\rho n^2 \\ H &= \mu-\rho n^2 \\ L &= M = C+A\sigma_1 = C+B\sigma_2 = \lambda+\mu \end{aligned}\right\}$$

Hence equation (11) has the roots

$$\left.\begin{aligned} q_1^2 &= 1-\frac{\rho n^2}{\lambda+2\mu} = 1-h^2 \\ q_2^2 &= 1-\frac{\rho n^2}{\mu} = 1-f^2 \end{aligned}\right\}$$

taking Timoshenko's notations. Then the frequency equation (13) is given by

$$4\sqrt{1-f^2}\sqrt{1-h^2}\tanh\left(\frac{2\pi c}{l}\sqrt{1-h^2}\right) = (2-h^2)^2\tanh\left(\frac{2\pi}{l}\sqrt{1-f^2}\right)$$

where l is the wavelength. This is Timoshenko's exact frequency equation for isotropic plate. If l be large we get the Rayleigh frequency equation,

$$16(1-f^2)(1-h^2) = (2-h^2)^4.$$

In particular when $\lambda = \mu$, $\frac{p}{\alpha} = (\cdot 9194)\sqrt{\frac{\mu}{\rho}}$.

It can now be shown that the solution $q^2 = 1$ of equation (10) is not a relevant one. A combination of $q^2 = 1$ and one of the other roots is not permissible. To see this, if we put $q_1 = 1$, so that from (14) $m_1 = -1$, the frequency equation (15) becomes an identity each side being equal to zero.

Inserting the values of m_1^{-1} and m_2^{-1} from (14) in the frequency equation (16), it is found that

$$\frac{(q_1^2-1)(q_2^2-1)}{\{(A-\rho n^2)-(A\sigma_1+C)-Cq_1^2\}\{(A-\rho n^2)-(A\sigma_1+C)-Cq_2^2\}}$$

can be removed from both sides of the equation. Therefore the frequency equation (16) for large wave lengths can be written as,

$$\begin{aligned}\{B(\rho n^2-A)+BCq_1^2+B\sigma_2(C+B\sigma_2)\}\{(\rho n^2-A)-B\sigma_2q_2^2\}q_1 \\ -\{B(\rho n^2-A+Cq_2^2)+B\sigma_2(C+B\sigma_2)\}\{(\rho n^2-A)-B\sigma_2q_1^2\}q_2 = 0, \qquad (17)\end{aligned}$$

since $C+B\sigma_2 = C+A\sigma_1$. Evidently the left-hand-side of (17) contains a factor q_1-q_2. Dividing by this factor, we get

$$\begin{aligned}B^2C\sigma_2q_1^2q_2^2-\{B^2\sigma_2(\rho n^2-A)+B^2\sigma_2^2(C+B\sigma_2)+BC(\rho n^2-A)\}q_1q_2 \\ -B^2\sigma_2(\rho n^2-A)(q_1^2+q_2^2)-BC(\rho n^2-A)^2-B\sigma_2(\rho n^2-A)(C+B\sigma_2) = 0. \qquad (18)\end{aligned}$$

Substituting for $q_1^2+q_2^2$, $q_1^2q_2^2$ and q_1q_2 we have after some reduction and writing ζ for ρn^2,

$$\zeta^2(\zeta-A)^2 = \frac{(\zeta-A)(\zeta-C)}{BC}B^2\{(\zeta-A)^2+B\sigma_2^2\}^2. \qquad (19)$$

Let

$$f(\zeta) = \zeta^2-\frac{B}{C}\frac{(\zeta-C)}{(\zeta-A)}\{(\zeta-A)^2+B\sigma_2^2\}^2.$$

When $\zeta = 0$, $f(\zeta)$ is negative. When $\zeta = C$, $f(\zeta)$ is positive. Hence there must be a real root between 0 and C. Taking the data for Stikka-spruce it is seen that the value of n^2 still remains approximately $(\cdot 9194)^2(C/\rho)$.

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A NOTE ON THE RATIO OF TWO NON-CENTRAL CHI-SQUARES

By

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Let x_i be a random variate normally distributed $N(\mu_i, 1)$. x_i's are independent among themselves, $i = 1, 2 \ldots m$. Also let y_j be a random variate normally distributed $N(\mu'_j, 1)$. y_j's are independent among themselves and are independent of x_i's. $j = 1, 2 \ldots n$.

Let

$$\lambda_1 = \sum \mu_i^2 \quad \text{and} \quad \lambda_2 = \sum \mu_j'^2.$$

Then

$$u = \sum_{i=1}^{m} x_i^2 \quad \text{and} \quad v = \sum_{j=1}^{n} y_j^2$$

will have non-central χ^2 distributions given by

$$p(u) = \frac{e^{-\lambda_1/2}}{2^{m/2}} \sum_{r=0}^{\infty} \frac{\lambda_1^{\,r}}{2^{2r} r!} \frac{u^{\frac{1}{2}m+r-1}}{\Gamma(\frac{1}{2}m+r)} \cdot e^{-\frac{1}{2}u}; \quad d.f. = m \tag{1}$$

and

$$p(v) = \frac{e^{-\lambda_2/2}}{2^{n/2}} \sum_{k=0}^{\infty} \frac{\lambda_2^{\,k}}{2^{2k} k!} \frac{v^{\frac{1}{2}n+k-1}}{\Gamma(\frac{1}{2}n+k)} \cdot e^{-\frac{1}{2}v}; \quad d.f. = n \tag{2}$$

Then the distribution of $F' = u/v$ will be given by

$$p(F') = e^{-(\lambda_1+\lambda_2)/2} \cdot \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda_1^{\,r} . \lambda_2^{\,k}}{2^r r! 2^k k!} \cdot \frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n+r+k)}{\Gamma(\frac{1}{2}m+r)\Gamma(\frac{1}{2}n+k)} \cdot \frac{F'^{\frac{1}{2}m+r-1}}{(1+F')^{\frac{1}{2}m+\frac{1}{2}n+r+k}} \tag{3}$$

This reduces to ordinary F distribution when $\lambda_1 = \lambda_2 = 0$. Let us define $E(x^{-s})$ as the s^{th} inverse moment if it exists. If ξ_1, and ξ_2 are two independent variates, then the s^{th} moment of $w = \xi_1/\xi_2$ will be the product of s^{th} moment of ξ^1 and the s^{th} inverse moment of ξ_2 if it exists. If the s^{th} inverse moment of ξ^2 does not exists then it follows that the s^{th} moment of w will not exist.

The p^{th} moment of $u = \sum x_i^2$ is given by

$$\mu'_p(u) = \frac{2^p . \Gamma(\frac{1}{2}m+p)}{\Gamma(m/2)} \cdot {}_1F_1(-p, \tfrac{1}{2}m, -\tfrac{1}{2}\lambda_1)$$

and the p^{th} inverse moment of $v = \sum y_j^2$ will be given by

$$\mu'_{-p}(v) = \frac{2^{-p} . \Gamma(\frac{1}{2}n-p)}{\Gamma(\frac{1}{2}n)} \cdot {}_1F_1(p, \tfrac{1}{2}n, -\tfrac{1}{2}\lambda_2)$$

Hence the p^{th} moment of $F' = u/v$ will be given by

$$\mu'_p(F') = \mu'_p(u) . \mu'_{-p}(v) = \frac{\Gamma(\frac{1}{2}m+p)\Gamma(\frac{1}{2}n-p)}{\Gamma(\frac{1}{2}m) . \Gamma(\frac{1}{2}n)} \cdot {}_1F_1(-p, \tfrac{1}{2}m, -\tfrac{1}{2}\lambda_1) . {}_1F_1(p, \tfrac{1}{2}n, -\tfrac{1}{2}\lambda_2). \tag{4}$$

This can also be obtained directly from (3) after certain simplifications.

Particular case:—If $\lambda_2 = 0$, then $v = \sum y_j^2$ will be a central χ^2. Then the distribution of $F = u/v$ will be given by

$$p(F) = \exp\left(-\tfrac{1}{2}\lambda_1\right) \sum_{r=0}^{\infty} \frac{\lambda_1^{\,r}}{2^r r!} \frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n+r)}{\Gamma(\frac{1}{2}m+r)\Gamma(\frac{1}{2}n)} \cdot \frac{(F)^{\frac{1}{2}m+r-1}}{(1+F)^{\frac{1}{2}m+\frac{1}{2}n+r}}$$

$$= \exp\left(-\tfrac{1}{2}\lambda_1\right). \frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n)}{\Gamma(\frac{1}{2}m).\Gamma(\frac{1}{2}n)} \cdot \frac{F^{\frac{1}{2}m-1}}{(1+F)^{\frac{1}{2}m+\frac{1}{2}n}} \cdot \sum_{r=0}^{\infty} \frac{(\frac{1}{2}m+\frac{1}{2}n)_r}{(\frac{1}{2}m)_r} . \frac{1}{r!} \left(\frac{\lambda}{2}\right)^r \frac{F^r}{(1+F)^r}$$

$$= \exp\left(-\tfrac{1}{2}\lambda_1\right) \cdot \frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n)}{\Gamma(\frac{1}{2}m+\frac{1}{2}n)} \cdot \frac{F^{\frac{1}{2}m-1}}{(1+F)^{\frac{1}{2}m+\frac{1}{2}n}} \cdot {}_1F_1\left(\frac{m}{2}+\frac{n}{2}, \frac{m}{2}, \frac{(\frac{1}{2}\lambda_1)F}{1+F}\right)$$

$$= \exp\left(-\frac{1}{2}\,\frac{\lambda_1}{(1+F)}\right). \frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n)}{\Gamma(\frac{1}{2}m).\Gamma(\frac{1}{2}n)} \cdot \frac{F^{\frac{1}{2}m-1}}{(1+F)^{\frac{1}{2}m+\frac{1}{2}n}} \cdot {}_1F_1\left(-\frac{n}{2}, \frac{m}{2}, \frac{-(\frac{1}{2}\lambda_1)F}{1+F}\right). \tag{5}$$

The k^{th} moment of this distributions can be shown to be

$$\mu'_k = \frac{\Gamma(\frac{1}{2}m+k)}{\Gamma(\frac{1}{2}m)} \cdot \frac{\Gamma(\frac{1}{2}n-k)}{\Gamma(\frac{1}{2}n)} \cdot {}_1F_1(-k.\ \tfrac{1}{2}m,\ -\tfrac{1}{2}\lambda_1). \tag{6}$$

The following recurrence relations between the moments μ'_k can be easily obtained from the recurrence relations between the moments of a non-central χ^2 distribution.

$$\left.\begin{aligned} \mu'_k &= \frac{n-2k-2}{m+2k} \cdot \mu'_{k+1} - \frac{n-2k-2}{(k+1)(m+2k)} \cdot \lambda_1 \frac{d\mu'_{k+1}}{d\lambda_1} \\ \mu'_{k+1} &= \left(\frac{\lambda_1}{2}\right)^{k+1} \frac{\Gamma(\frac{1}{2}n-k)}{\Gamma(\frac{1}{2}n)} + \frac{(m+2k)(k+1)}{(n-2k-2)} \cdot \lambda_1^{k+1} \int_{\lambda_1}^{\infty} \frac{\mu'_k}{\lambda_1^{\,k+2}}\, d\lambda_1 \end{aligned}\right\} \tag{7}$$

It is easy to calculate all the moments of this distribution with the help of (7).

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ADVANCEMENT OF FLUID OVER AN INFINITE PLATE

By

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Part I Boundary layer flow

Introduction

An infinite plate defined by the equation $y = 0$ is placed on the zx-plane. Axis of x is taken along the length of the plate and the axis of y perpendicular to it. At the time $t = 0$ the fluid occupies a vast expanse of space, for which, $x \leq 0$, and the front surface of the fluid, the plane $x = 0$, starts moving with a constant velocity U_0 in the x-direction. After an interval of time t the stream has advanced a distance $x = U_0 t$ along the plate. The front surface at any instant is defined by the equation $x = U_0 t$, which is always kept moving with a constant velocity U_0. The components of velocity are non-existent for $x > U_0 t$, because the column of fluid has not advanced so far.

Equation of motion

The boundary layer equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

where u is the component of velocity along the plate, v is the component of velocity perpendicular to the plate and ν is the kinematic coefficient of viscosity.

We introduce the stream function

$$\psi = \sqrt{\{\nu U_0(U_0 t - x)\}} f(\eta) \tag{3}$$

where

$$\eta = y \Big/ \sqrt{\left\{\frac{\nu}{U_0}(U_0 t - x)\right\}}.$$

Therefore

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x};$$

$$u = U_0 f'(\eta) \tag{4}$$

$$v = \frac{1}{2}\left(\frac{\nu U_0}{U_0 t - x}\right)^{\frac{1}{2}} [f(\eta) - \eta f'(\eta)] \tag{5}$$

Making these substitutions (1) becomes

$$2f''' - ff'' + \eta f'' = 0 \tag{6}$$

where dashes denote differentiation with respect to η. The boundary conditions are

$$u = v = 0 \quad \text{on} \quad y = 0, \quad x \leq U_0 t$$

$$i.e. \quad f(0) = f'(0) = 0$$

$$\text{and} \quad u = U_0 \quad \text{for} \quad y = \infty, \quad x \leq U_0 t$$

$$i.e. \quad f'(\infty) = 1$$

$$\text{and} \quad u = u_0 \quad \text{for} \quad x = u_0 t, \quad \text{for} \quad y > 0.$$

We solve equation (6) by the method of successive approximations.

In (6) we put $\eta = 2z$, the equation becomes

$$f''' + 2zf'' - ff'' = 0. \tag{7}$$

We first solve the equation

$$f_1''' + 2zf_1'' = 0 \tag{8}$$

where f_1 represents the first approximate solution.

The solution of this equation which makes

$$f_1(0) = f_1'(0) = 0 \quad \text{and} \quad f_1'(\infty) = 2 \quad \text{is}$$

$$f_1 = 2z \operatorname{erf} z + \frac{2}{\sqrt{\pi}} (e^{-z^2} - 1) \tag{9}$$

As a second approximation we put

$$f = f_1 + f_2$$

and we are content to solve the differential equation

$$f_2''' + 2zf_2'' = f_1 f_1'' \tag{10}$$

or

$$f_2''' + 2zf_2'' = \frac{4}{\sqrt{\pi}} \left[ze^{-z^2} \operatorname{erf} z + \frac{1}{\sqrt{\pi}} (e^{-2z^2} - e^{-z^2}) \right]. \tag{11}$$

The boundary conditions are

$$f_2'(0) = f_2'(\infty) = 0$$

$$f(0) = 0$$

The solution of (11) satisfying these conditions is

$$f_2 = \frac{4}{\sqrt{\pi}} \left[\int_0^z \left[\frac{\sqrt{\pi}}{8} (\operatorname{erf} z)^2 - \tfrac{1}{4} ze^{-z^2} \operatorname{erf} z - \frac{1}{4\sqrt{\pi}} e^{-2z^2} \right.\right.$$
$$\left.\left. + \frac{1}{2\sqrt{\pi}} e^{-z^2} + \frac{2-\pi}{8\sqrt{\pi}} \operatorname{erf} z - \frac{1}{4\sqrt{\pi}} \right] dz. \tag{12}$$

The second approximate solution is

$$f = 2z \operatorname{erf} z + \frac{2}{\sqrt{\pi}} (e^{-z^2} - 1) + \frac{4}{\sqrt{\pi}} \int_0^z \left[\frac{\sqrt{\pi}}{8} (\operatorname{erf} z)^2 - \tfrac{1}{4} ze^{-z^2} \operatorname{erf} z \right.$$
$$\left. - \frac{1}{4\sqrt{\pi}} e^{-2z^2} + \frac{1}{2\sqrt{\pi}} e^{-z^2} + \frac{2-\pi}{8\sqrt{\pi}} \operatorname{erf} z - \frac{1}{4\sqrt{\pi}} \right] dz. \tag{13}$$

Substituting in (4) and (5) we get the components of velocity.

Part II. Turbulent flow.

The equation of motion on the 'Momentum Transfer theory' of Prandtl and the 'Vorticity Transport theory' of Taylor are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left[l_1^2\left(\frac{\partial u}{\partial y}\right)^2\right] \qquad P(1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=l^2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}. \qquad T(1)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2)$$

We introduce the stream function defined by the relation

$$\psi=U_0\sqrt{(U_0t-x)}f(\eta)$$

where

$$\eta=\frac{y}{\sqrt{(U_0t-x)}}$$

The components of velocity are

$$u=\frac{\partial\psi}{\partial y},\qquad v=-\frac{\partial\psi}{\partial x}.$$

Therefore

$$u=U_0f'(\eta) \qquad (3)$$

$$v=\frac{U_0}{2\sqrt{(U_0t-x)}}\left[f(\eta)-\eta f'(\eta)\right] \qquad (4)$$

Assume

$$2l_1^2=l^2=\frac{1}{2c^3}\sqrt{(U_0t-x)} \qquad (5)$$

where c is a positive unknown constant. Both P(1) and T(1) reduce to the ssmc form:

$$f''(f'''-c^3f+c^3\eta)=0 \qquad (6)$$

where dashes denote differentiation with respect to η. The solution of (6) which makes

$$u=v=0 \quad \text{on} \quad y=0, \quad x\leq U_0t$$

$$i.e. \quad f(0)=f'(0)=0$$

$$\text{and} \quad U=u_0 \quad \text{for} \quad y=\infty, \quad x\leq U_0t$$

$$i.e. \quad f'(\infty)=1$$

$$\text{and} \quad u=U_0 \quad \text{for} \quad x=U_0t, \quad \text{for } y>0 \quad \text{is}$$

$$f(\eta)=\eta-\frac{2}{\sqrt{3c}}e^{-\frac{1}{2}c\eta}\sin\sqrt{\tfrac{3}{2}}\eta. \qquad (7)$$

The components of velocity are given by

$$u=U_0\left[1-\frac{2}{\sqrt{3}}e^{-\frac{1}{2}c\eta}\sin\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}c\eta\right)\right]$$

$$v = \frac{U_0 e^{-\frac{1}{2}c\eta}}{\sqrt{3c}\sqrt{(U_0 t - x)}} \left[c\eta \sin\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} c\eta\right) - \sin \frac{\sqrt{3}}{2} c\eta \right]$$

These formulae reveal that velocity is non-existent for $x > U_0 t$. The unknown constant c may be found by noting the velocity at any point in the turbulent region.

In conclusion I thank Prof. M. Ray, D.Sc. for his kind help in the preparation of this note.

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SOME PROPERTIES OF GENERALISED HANKEL TRANSFORM

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1. **Introduction.** Recently*, I have given a generalisation of the wellknown Hankel transform by considering the Integral equation

$$f(x) = (\tfrac{1}{2})^{\lambda} \int_0^{\infty} (xy)^{\lambda+\frac{1}{2}} J_{\lambda}^{\mu}(\tfrac{1}{4}x^2y^2) g(y) dy \tag{1.1}$$

where, $J_{\mu}^{\lambda}(z)$ is Wright's (1934) generalised Bessel function derived by the series

$$J_{\lambda}^{\mu}(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!\Gamma(1+\lambda+\mu r)}, \quad \mu > 0.$$

In the particular case when $\mu = 1$, (1.1) yields the known Hankel transform. I have also given an Inversion formula for (1.1), *viz.*,

$$g(x) = \frac{2^{2-2/\mu-2\lambda/\mu+\lambda}}{\mu} \int_0^{\infty} (xy)^{-(2/\mu)(\frac{1}{3}\mu-1+\lambda(\frac{1}{2}\mu-1))} J_{1/\mu+\lambda/\mu-1}^{1/\mu}\left[\left(\frac{x^2y^2}{4}\right)^{1/\mu}\right] f(y) dy, \tag{1.2}$$

where $\mu > 0$, $Re\ \lambda > 1$ and $f(x)$ and $g(x)$ fulfil suitable conditions to make integrals in (1.1) and (1.2) converge.

The object of the present paper is to develop the theory of this generalised transform. In §§2-6 I have proved five theorems for this transform which are exactly analogous to the corresponding results in the theory of the ordinary transform but have a generalised appearance. Lastly, it is interesting to note that there is no analogue of the theorem proved in §7, in the ordinary Hankel transform theory.

Another interesting case of this transform which has been considered by me is for $\mu = \frac{1}{2}$. The kernel

$$\left(\frac{1}{2}\right)^{\lambda} x^{\lambda+\frac{1}{2}} J_{\lambda}^{\mu}\left(\frac{x^2}{4}\right)$$

reduces to

$$(\tfrac{1}{2})^{\lambda} x^{\lambda+\frac{1}{2}} \sqrt{\pi} \left\{\left(\frac{-4}{x^{4/3}}\right)^{\lambda-\frac{1}{2}} J_{\lambda,-\frac{1}{2}}(-\tfrac{3}{4}x^{4/3}) - \frac{x^2}{8}\left(-\frac{4}{x^{4/3}}\right)^{\lambda+1} J_{\lambda+\frac{1}{2},\frac{1}{2}}(-\tfrac{3}{4}x^{1/3})\right\}$$

and the kernel in (1.2) reduces to

$$\sqrt{\pi}\, 2^{-1-\lambda} x^{\lambda/3+11/6} J_{\lambda,\lambda+\frac{1}{2}}(\tfrac{3}{4}x^{4/3})$$

where $J_{m,n}(x)$ is Humbert's function, defined by the series

$$J_{m,n}(x) = \frac{(\frac{1}{3}x)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\, {}_0F_2(m+1, n+1; -x^3/27).$$

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* *Cf.* Thesis on "The Theory of Hankel transform and Self-reciprocal functions," approved for the Ph.D. degree of the University of Lucknow, 1950. See also—Agarwal (1950).

We shall, henceforth call two functions $f(x)$ and $g(x)$ connected by the relation (1.1) to be the J^μ_λ-transforms or the generalised Hankel transforms. In the case when $\mu = 1$, we shall call them the Humbert transforms or simply Humbert pairs.

2. Theorem 1. *If $f(x)$ and $g(x)$ are J^μ_λ-transforms, then*

$$\int_0^\infty \frac{f(x)}{x^{\frac{1}{2}}}\,dx = \frac{\Gamma(\frac{1}{2}\lambda+\frac{1}{2})}{\Gamma(1+\lambda-\frac{1}{2}\mu-\frac{1}{2}\mu\lambda)}\int_0^\infty \frac{g(x)}{x^{\frac{1}{2}}}\,dx \tag{2.1}$$

provided the integrals converge and $0<\mu\leq 1$, $Re\lambda>-1$.

Proof. We have

$$f(x) = (\tfrac{1}{2})^\lambda \int_0^\infty (xy)^{\lambda+\frac{1}{2}} J^\mu_\lambda\left(\frac{x^2y^2}{4}\right) g(y)dy$$

Therefore

$$\int_0^\infty \frac{f(x)}{x^{\frac{1}{2}}}\,dx = (\tfrac{1}{2})^\lambda \int_0^\infty dx x^\lambda \int_0^\infty y^{\lambda+\frac{1}{2}} J^\mu_\lambda\left(\frac{x^2y^2}{4}\right) g(y)dy$$

If the change in the order of integration be permissible, we get

$$\int_0^\infty \frac{f(x)}{x^{\frac{1}{2}}}\,dx = (\tfrac{1}{2})^\lambda \int_0^\infty y^{\lambda+\frac{1}{2}} g(y)dy \int_0^\infty x^\lambda J^\mu_\lambda\left(\frac{x^2y^2}{4}\right)dx. \tag{2.2}$$

Integrating with the help of an integral by Gupta (1948)

$$\int_0^\infty s^\nu J^\mu_\lambda(sw)ds = \frac{\Gamma(1+\nu)}{\Gamma(1+\lambda-\mu-\mu\nu)w^{\nu+1}}, \tag{2.2a}$$

valid by the principle of analytic continuation when $\mu\leq 1$ and $Re\,\nu>-1$ with an additional condition $Re(\lambda-2\nu)>\frac{1}{2}$ in case $\mu = 1$, we have

$$\int_0^\infty \frac{f(x)}{x^{\frac{1}{2}}}\,dx = \frac{\Gamma(\frac{1}{2}\lambda+\frac{1}{2})}{\Gamma(1+\lambda-\frac{1}{2}\mu\overline{1+\lambda})}\int_0^\infty \frac{g(y)}{y^{\frac{1}{2}}}\,dy \tag{2.3}$$

which proves the theorem.

The change in the order of integration in (2.2) is permissible if the integrals in (2.1) exist and

$$\int_0^\infty y^{\lambda+\frac{1}{2}} J^\mu_\lambda\left(\frac{x^2y^2}{4}\right) g(y)dy \tag{2.4}$$

and

$$\int_0^\infty x^\lambda J^\mu_\lambda\left(\frac{x^2y^2}{4}\right)dx \tag{2.5}$$

are absolutely convergent. The integral (2.4) being assumed to be absolutely convergent, the integral (2.5) is absolutely convergent for $Re\lambda \geq -1$ and $0 \leq \mu \leq 1$ since

$$J^\mu_\lambda(x) \sim O\left[x^{-k(\frac{1}{2}+\lambda)}\exp\left\{\frac{(\mu x)^k}{\mu k}\cos\pi k\right\}, \; k = 1/(\mu+1)\right.$$

for large values of x and

$$J^\mu_\lambda(x) \sim O(1)$$

for small values of x.

Applying the principle of analytic continuation our result is true for $0 < \mu \leq 1$, $Re\lambda > -1$ when the integrals in (2.3) have a meaning.

As a verification of our theorem let us consider the Humbert pair

$$g(x) = \frac{1}{x^2} e^{1/8x^4} D_{-\frac{1}{2}3\lambda - 9/4}\left(\frac{1}{\sqrt{2x^2}}\right)$$

and

$$f(x) = \frac{2^{\frac{1}{2}\lambda - 1/8} x^{\frac{1}{2}\lambda + 3/4}}{\Gamma(\frac{1}{2}3\lambda + \frac{9}{4})} K_{-\frac{1}{2}\lambda + \frac{1}{4}}(x) \quad [Re\lambda > -1]$$

Then

$$\int_0^\infty \frac{f(x)}{\lambda^{\frac{1}{2}}} dx = \frac{2^{\frac{1}{2}\lambda - 1/8}}{\Gamma(\frac{1}{2}3\lambda + \frac{9}{4})} \int_0^\infty x^{\frac{1}{2}\lambda + \frac{1}{4}} K_{-\frac{1}{2}\lambda + \frac{1}{4}}(x) dx$$

Integrating with the help of the integral (Watson, 1944), we get,

$$\int_0^\infty \frac{f(x)}{x^{\frac{1}{2}}} dx = 2^{\frac{1}{2}3\lambda - 7/8} \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2}\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2}3\lambda + \frac{9}{4})} \quad (Re\lambda > -1).$$

Also evaluting the integral

$$\int_0^\infty \frac{g(x)}{x^{\frac{1}{2}}} dx = \int_0^\infty x^{-5/2} \exp\left(\frac{1}{8x^4}\right) D_{-\frac{1}{2}3\lambda - 9/4}\left(\frac{1}{\sqrt{2x^2}}\right) dx$$

by the help of

$$\int_0^\infty x^{s-1} e^{\frac{1}{4}x^2} D_n(x) dx = \frac{\Gamma(s)\Gamma(-\frac{1}{2}s - \frac{1}{2}n)}{2^{\frac{1}{2}n + \frac{1}{2}s + 1}\Gamma(-n)},$$

we obtain

$$\int_0^\infty \frac{g(x)}{x^{\frac{1}{2}}} dx = 2^{\frac{1}{2}3\lambda - 7/8} \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2}3\lambda + \frac{3}{4})}{\Gamma(\frac{1}{2}3\lambda + \frac{9}{4})}$$

which verifies our theorem.

3. Theorem 2. *If $\phi_1(x)$ and $\phi_2(x)$ are J^μ_λ-transforms of $\psi_1(x)$ and $\psi_2(x)$ respectively,*

$$\int_0^\infty \phi_1(x)\psi_2(x)dx = \int_0^\infty \phi_2(x)\psi_1(x)dx \tag{3.0}$$

provided the integrals exist.

This may be called the Parseval theorem for our generalised Hankel Transform.

Proof. We have

$$\phi_1(x) = (\tfrac{1}{2})^\lambda \int_0^\infty (xy)^{\lambda + \frac{1}{2}} J^\mu_\lambda\left(\frac{x^2y^2}{4}\right) \psi_1(y) dy \tag{3.1}$$

and

$$\phi_2(x) = (\tfrac{1}{2})^\lambda \int_0^\infty (xy)^{\lambda + \frac{1}{2}} J^\mu_\lambda\left(\frac{x^2y^2}{4}\right) \psi_2(y) dy, \tag{3.2}$$

Therefore,

$$\int_0^\infty \phi_1(x)\psi_2(x)dx = \int_0^\infty \psi_2(x)dx.(\tfrac{1}{2})^\lambda \int_0^\infty (xy)^{\lambda + \frac{1}{2}} J^\mu_\lambda\left(\frac{x^2y^2}{4}\right) \psi_1(y) dy.$$

Changing the order of integration,

$$\int_0^\infty \phi_1(x)\psi_2(x)dx = (\tfrac{1}{2})^\lambda \int_0^\infty \psi_1(y)dy \int_0^\infty (xy)^{\lambda+\frac{1}{2}} J_\lambda^\mu\left(\frac{x^2y^2}{4}\right)\psi_2(x)dx = \int_0^\infty \psi_1(y)\phi_2(y)dy$$

which proves the result.

Assuming that the integrals (3.1) and (3.2) are absolutely convergent the change will therefore be justified if the integrals (3.0) are absolutely convergent.

Example I: Let us consider the J_λ^μ-transforms*

$$\phi_1(x) = x^{\frac{1}{2}\lambda+\frac{1}{2}} K_{\frac{1}{2}\lambda}(x)$$

$$\psi_1(x) = 2^{\frac{1}{2}\lambda} x^{-3/2} \sum_{n=0}^\infty \frac{(-)^n \Gamma(\mu n + 1 + \lambda - \frac{1}{2}\mu\lambda)}{n!x^{2n}} \quad (0 < \mu \leq 1,\ Re\lambda > -1)$$

and

$$\phi_2(x) = \frac{1}{2^{\lambda+1}} x^{\lambda+\frac{1}{2}} \sum_{r=0}^\infty \frac{\Gamma(r+\frac{1}{2}\lambda+\frac{3}{4})}{r!\Gamma(1+\lambda+\mu r)}\left(\frac{-x^2}{4}\right)^r$$

$$\psi_2(x) = e^{-x^2} \quad (0 < \mu \leq 1,\ Re\lambda > -\tfrac{3}{2}).$$

Then our theorem gives

$$\int_0^\infty x^{\frac{1}{2}\lambda+\frac{1}{2}} e^{-x^2} K_{\frac{1}{2}\lambda}(x)dx = 2^{-\frac{1}{2}\lambda-1}\int_0^\infty x^{\lambda-1} \sum_{r=0}^\infty \frac{\Gamma(r+\frac{1}{2}\lambda+\frac{3}{4})}{r!\Gamma(1+\lambda+\mu\gamma)}\left(-\frac{x^2}{4}\right)^r$$
$$\times \sum_{n=0}^\infty \frac{(-)^n\Gamma(\mu n+1+\lambda-\frac{1}{2}\mu\lambda)}{n!x^{2n}}\,dx$$

provided both the integrals exist.

Since

$$K_\lambda(x) = O(x^{-\frac{1}{2}}e^{-x}), \quad \text{for large } x,$$
$$= O(ax^{-\lambda}+bx^\lambda), \quad \text{for small } x,$$

the left hand integral converges for $-1 < Re\lambda < 3$.

Also following Wright (1940) we find that

$$\sum_{n=0}^\infty \frac{(-)^n\Gamma(1+\lambda+\mu n-\frac{1}{2}\mu\lambda)}{n!x^{2n}} = O(1) \quad \text{for large } x.$$
$$= O(x^{\frac{1}{\mu}(1+\lambda-\frac{1}{2}\mu\lambda)}), \quad \text{for small } x.$$

Also

$$\sum_{r=0}^\infty \frac{\Gamma(r+\frac{3}{4}+\frac{1}{2}\lambda)}{r!\Gamma(1+\lambda+\mu r)}\left(\frac{-x^2}{4}\right)^r = O(1), \quad \text{for small } x.$$
$$= O(x^{-\lambda-3/2}), \quad \text{for large } x.$$

Hence, the integral on the right converges for $Re\lambda > -1$, $0 < \mu \leq 1$.

Now, the integral

$$\int_0^\infty x^{\frac{1}{2}\lambda+\frac{1}{2}} e^{-x^2} K_{\frac{1}{2}\lambda}(x)dx$$

can be evaluated with the help of the integral (Meijer, 1934)

$$W_{k,m}(\zeta^2) = \frac{4\zeta e^{-\frac{1}{2}\zeta^2}}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\int_0^\infty e^{-u^2}K_{2m}(2\zeta u)u^{-2k}du, \ [Re(\tfrac{1}{2}\pm m-k)>0].$$

* *Cf* Thesis *loc. cit.*

which gives

$$\int_0^\infty x^{\frac{1}{2}\lambda+\frac{1}{4}} e^{-x^2} K_{\frac{1}{2}\lambda}(x)dx = \frac{1}{2}\,\frac{e^{1/8}}{\{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4}+\frac{1}{2}\lambda)\}^{-1}}\,W_{-\frac{1}{4}\lambda-\frac{1}{4},\frac{1}{4}\lambda}(\tfrac{1}{4})$$

Therefore,

$$\int_0^\infty x^{\lambda-1}\sum_{r=0}^\infty \frac{(-)^r\Gamma(1+\lambda-\frac{1}{2}\mu\lambda+\mu r)}{r!\;x^{2r}}\sum_{r=0}^\infty \frac{\Gamma(\frac{1}{2}\lambda+r+\frac{3}{4})(-\frac{1}{4}x^2)^r}{r!\;\Gamma(1+\lambda+\mu r)}\,dx$$

$$= \frac{2^{\frac{1}{2}\lambda}e^{1/8}}{\{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4}+\frac{1}{2}\lambda)\}^{-1}}\,W_{-\frac{1}{4}\lambda-\frac{1}{4},\frac{1}{4}\lambda}(\tfrac{1}{4}).$$

the integral being convergent under the conditions stated above.

Example 2 Let us cosider the Humbert pair*

$$\phi_1(x) = \frac{1}{2^\lambda}x^{\lambda+\frac{1}{2}}J_\rho(x)K_\rho(x)$$

and

$$\psi_1(x) = \frac{\Gamma(1+\lambda+\frac{1}{2}\rho)}{\Gamma(1+\rho)}\exp\left(-\frac{1}{2x^4}\right)y^{-\lambda+\frac{1}{2}}M_{\lambda+\frac{1}{2},\rho}\left(\frac{1}{x^4}\right)$$

$$\phi_2(x) = \frac{2^{\frac{1}{2}\lambda-1/8}}{\Gamma(\frac{1}{2}3\lambda+\frac{9}{4})}\,x^{\frac{1}{2}\lambda+3/4}K_{-\frac{1}{2}\lambda+\frac{1}{4}}(x)$$

and

$$\psi_2(x) = \frac{1}{x^2}\exp\left(\frac{1}{8x^4}\right)D_{-\frac{1}{2}3\lambda-9/4}\left(\frac{1}{\sqrt{2x^2}}\right)\;[Re\lambda>-1].$$

Our theorem now gives

$$\frac{1}{2^\lambda}\int_0^\infty x^{\lambda-3/2}J_\rho(x)K_\rho(x)D_{-\frac{1}{2}3\lambda-9/4}\left(\frac{1}{\sqrt{2x^2}}\right)\exp\left(\frac{1}{8x^4}\right)dx$$

$$= \frac{2^{\frac{1}{2}\lambda-1/8}\Gamma(1+\lambda+\frac{1}{2}\rho)}{\Gamma(1+\rho)\Gamma(\frac{1}{2}3\lambda+\frac{9}{4})}\int_0^\infty y^{-\frac{1}{2}\lambda+5/4}\exp\left(-\frac{1}{2y^4}\right)M_{\lambda+\frac{1}{2},\rho}\left(\frac{1}{y^4}\right)K_{-\frac{1}{2}\lambda+\frac{1}{4}}(y)dy$$

Let us evaluate the right hand integral

$$I = \int_0^\infty y^{-\frac{1}{2}\lambda+5/4}\exp\left(-\frac{1}{2y^4}\right)M_{\lambda+\frac{1}{2},\rho}\left(\frac{1}{y^4}\right)K_{-\frac{1}{2}\lambda+\frac{1}{4}}(y)dy$$

Substituting the Barnes's type of integral for $K_\nu(y)$ (Watson, 1944, p, 388)

$$K_\nu(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s-2}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right)y^{-s}ds,\quad [\text{Re}\; s>R\,|\,\nu\,|]$$

and inverting the order of integration which can be easily justified, we get

$$I = \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} 2^s\Gamma(\tfrac{1}{2}s+\tfrac{1}{4}\lambda-\tfrac{1}{8})\Gamma(\tfrac{1}{2}s-\tfrac{1}{4}\lambda+\tfrac{1}{8})ds\int_0^\infty y^{-s+\frac{1}{2}\lambda+5/4}\exp\left(-\frac{1}{2y^4}\right)M_{\lambda+\frac{1}{2},\rho}\left(\frac{1}{y^4}\right)dy$$

$$= \frac{1}{32\pi i}\int_{c-i\infty}^{c+i\infty} 2^s\Gamma(\tfrac{1}{2}s+\tfrac{1}{4}\lambda-\tfrac{1}{8})\Gamma(\tfrac{1}{2}s-\tfrac{1}{4}\lambda+\tfrac{1}{8})ds\int_0^\infty z^{\frac{1}{4}s+\lambda/8+17/16}e^{-z}\times{}_1F_1(\rho-\lambda;\,2\rho+1;\,z)dz$$

* Thesis, *loc. cit.*

Expanding the ${}_1F_1$-function and integrating term by term, a process justified due to the uniform convergence of the ${}_1F_1$-function and using the formula

$$ {}_2F_1(a,\, b\,,\, c\,;\, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad [Re\,(c-a-b) > 0] $$

we get

$$ \frac{1}{32\pi i}\,\frac{\Gamma(1+2\rho)}{\Gamma(1+\rho+\lambda)}\int_{c-i\infty}^{c+i\infty} \frac{2^{s}\Gamma(\frac{1}{2}s+\frac{1}{4}\lambda-\frac{1}{8})\Gamma(\frac{1}{2}s-\frac{1}{4}\lambda+\frac{1}{8})\Gamma(\frac{1}{2}s+\frac{1}{8}\lambda-\frac{1}{16})\Gamma(\frac{17}{16}-\frac{1}{4}s+\frac{7}{8}\lambda)}{\Gamma(-\frac{s}{4}+\frac{17}{16}-\frac{1}{8}\lambda)}\, ds. $$

Evaluating the residues at the poles of the integrand, we get

$$ \int_0^\infty x^{\lambda-3/2} J_\rho(x) K_\rho(x) D_{-\frac{1}{4}3\lambda-9/4}\left(\frac{1}{\sqrt{2x^2}}\right)\exp\left(\frac{1}{8x^4}\right) dx $$

$$ = \frac{2^{\frac{1}{4}5\lambda-83/8}\Gamma(1+\lambda+\frac{1}{2}\rho)\Gamma(1+2\rho)}{\Gamma(1+\rho)\Gamma(\frac{1}{2}3\lambda+\frac{9}{4})\Gamma(1+\rho+\lambda)}\Bigg[\sum_{n=0}^{\infty} 2^{-\frac{1}{4}\lambda+5/4}\,\frac{\Gamma(-n+\frac{1}{4}-\frac{1}{2}\lambda)\Gamma(\lambda+\frac{1}{2}n+1)}{n\,!\,\Gamma(\rho+1+\frac{1}{2}n)}\,\Gamma(-\tfrac{1}{3}n+\rho)(-\tfrac{1}{4})^n $$

$$ +\sum_{n=0}^{\infty} 2^{-\frac{1}{4}\lambda-4\rho+9/4}\,\frac{\Gamma(-2n-2\rho)\Gamma(-2n-\frac{1}{2}\lambda+\frac{1}{4}-2\rho)\Gamma(n+\lambda+\rho+1)}{n\,!\,\Gamma(2\rho+1+n)}\,(-\tfrac{1}{16})^n $$

$$ +\sum_{n=0}^{\infty} 2^{\frac{1}{4}\lambda+9/4}\,\frac{\Gamma(-n+\frac{1}{2}\lambda-\frac{1}{4})\Gamma(-\frac{1}{2}n+\frac{1}{4}\lambda-\frac{1}{8})\Gamma(\frac{9}{8}+\frac{1}{2}n+\frac{1}{4}3\lambda)}{n\,!\,\Gamma(\rho+\frac{1}{2}n-\frac{1}{4}\lambda+\frac{9}{8})}\,(-\tfrac{1}{4})^n\Bigg] $$

valid for $Re\lambda > -\frac{3}{2}$, $Re(2\lambda+\rho+\frac{3}{2}) > 0$.

4. Theorem 3. *If $f(x)$ and $g(x)$ are J_λ^μ-transforms and*

$$ g(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \chi(s) y^{-s} ds \tag{4.1} $$

where, for

$$ s = \sigma + it $$

$$ \chi(s) = O(e^{-k|t|}), \quad k > 0, \text{ and } |t| \to \infty $$

and σ is a value of c in the strip $0 < c < 1$, then,

$$ f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2^{s-\frac{1}{2}}\Gamma(\frac{1}{2}\lambda+\frac{1}{4}+\frac{1}{2}s)}{\Gamma[1-\frac{1}{4}\mu+\lambda(1-\frac{1}{2}\mu)-\frac{1}{2}\mu s]}\,\chi(1-s) x^{-s} x\, ds \tag{4.2} $$

valid under the conditions

$$ 0 < \mu \leq 1, \quad Re(\lambda - s + \tfrac{3}{2}) > 0. $$

Proof. From (1.1), we have

$$ f(x) = (\tfrac{1}{2})^\lambda \int_0^\infty (xy)^{\lambda+\frac{1}{2}} J_\lambda^\mu\left(\frac{x^2y^2}{4}\right) g(y) dy. $$

Substituting the value of $g(y)$ from (4.1) we get.

$$ f(x) = \left(\frac{1}{2}\right)^\lambda \frac{1}{2\pi i}\int_0^\infty (xy)^{\lambda+\frac{1}{2}} J_\lambda^\mu\left(\frac{x^2y^2}{4}\right) dy \int_{c-i\infty}^{c+i\infty} \chi(s) y^{-s} ds \tag{4.3} $$

If the change in the order of integration be permissible then,

$$f(x) = \left(\frac{1}{2}\right)^{\lambda} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \chi(s) ds \int_0^{\infty} y^{\lambda-s+\frac{1}{2}} J_{\lambda}^{\mu}\left(\frac{x^2y^2}{4}\right) dy$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\lambda + \frac{1}{4} + \frac{1}{2}s)\chi(1-s)x^{-s}}{\Gamma(1 - \frac{1}{4}\mu + \lambda 1 - \frac{1}{2}\mu - \frac{1}{2}\mu s)} \chi(1-s)x^{-s}ds$$

which proves our result.

To justify the change in the order of integration in (4.4) we see that both the s- and the y- integrals in (4.4) converge absolutely for $Re(\lambda - s + \frac{3}{2}) > 0$ and $0 < \mu \leq 1$. The repeated integral also exists since

$$\chi(s) = O(e^{-k|t|}), \quad k > 0 \text{ as } |t| \to \infty.$$

Theorem 3(a). *If we take* $\mu = 1$ *and*

$$\chi(s) = 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4} + \tfrac{1}{2}\lambda + \tfrac{1}{2}s)\psi(s)$$

where $\psi(s) = \psi(1-s)$ *in the strip* $0 < c < 1$ *then we have*

$$f(x) = g(x) \ \ i.e. \ \ f(x) \text{ is } R_{\lambda}.$$

Theorem 3(b). *If*

$$g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4} + \tfrac{1}{2}\lambda + \tfrac{1}{2}s)\psi(s)x^{-s}xds$$

i.e., it is R_{λ} *where,*

$$\psi(s) = \psi(1-s)$$

in the strip $0 < c < 1$, *and* $f(x)$ *is its* J_{λ}^{μ}*-transform then*

$$F(x) = -\frac{d}{dx}\left(x^2 \frac{d}{dx} f(x)\right).$$

is also the J_{λ}^{μ}*-transform of*

$$G(x) = -\frac{d}{dx}\left(x^2 \frac{d}{dx} g(x)\right)$$

Example 1. Let us take

$$\chi(s) = 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}\lambda + \tfrac{1}{2}s + \tfrac{1}{4}), \text{ in } (4.1)$$

Then (Titchmarsh, 1937, p. 260), $g(x) = 2^{3/4-\frac{1}{2}\lambda}x^{\lambda+\frac{1}{2}}e^{-\frac{1}{2}x^2}$.

Our theorem gives

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{s-\frac{1}{2}}\Gamma(\frac{1}{4} + \frac{1}{2}\lambda + \frac{1}{2}s) \, 2^{\frac{1}{2}-\frac{1}{2}s}\Gamma(\frac{1}{2}\lambda - \frac{1}{2}s + \frac{3}{4})}{\Gamma(1 - \frac{1}{4}\mu + \lambda 1 - \frac{1}{2}\mu - \frac{1}{2}\mu s)} x^{-s}ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{\frac{1}{2}s}\Gamma(\frac{1}{4} + \frac{1}{2}\lambda + \frac{1}{2}s)\Gamma(\frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}s)}{\Gamma(1 - \frac{1}{4}\mu + \lambda 1 - \frac{1}{2}\mu - \frac{1}{2}\mu s)} x^{-s}ds$$

$$= 2^{3/4-\frac{1}{2}\lambda}x^{\lambda+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{\Gamma(\lambda+1+r)}{r!\,\Gamma(\lambda+1+\mu r)} \left(\frac{-x^2}{2}\right)^r.$$

In particular for $\mu = 1$, we get

$$f(x) = 2^{3/4-\frac{1}{2}\lambda}x^{\lambda+\frac{1}{2}}e^{-\frac{1}{2}x^2}, \quad \text{a } R_{\lambda}^{\mu} \text{ function.}$$

Also for $\mu = \frac{1}{2}$, we get the Humbert pair

$$g(x) = 2^{3/4-\frac{1}{2}\lambda}x^{\lambda+\frac{1}{2}}e^{-\frac{1}{2}x^2}$$

and

$$f(x) = 2^{3/4+\frac{1}{2}\lambda}x^{\lambda+\frac{1}{2}}\{{}_2f_2(\tfrac{1}{2}\lambda+\tfrac{1}{4},\ \tfrac{1}{2}\lambda+1;\ 1+\lambda,\ \tfrac{1}{2};\ \tfrac{1}{4}x^4)-\tfrac{1}{4}x^2{}_2f_2(\tfrac{1}{2}\lambda+1,\ \tfrac{1}{2}\lambda+\tfrac{3}{4};\ \tfrac{1}{2}\lambda+\tfrac{3}{2},\ \tfrac{3}{2};\ \tfrac{1}{4}x^4)\}$$

Example 2. Taking

$$\chi(s) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}s)\Gamma(\frac{3}{4}+\frac{1}{2}\beta-\frac{1}{2}s)}{2\Gamma(1+\frac{1}{2}\alpha+\frac{1}{2}\beta)}.$$

Then (Titchmarsh, 1937, p. 270)

$$g(x) = \frac{x^{\beta+\frac{1}{2}}}{(1+x^2)^{\frac{1}{2}\alpha+\frac{1}{2}\beta+1}}$$

and therefore we obtain from (4.2)

$$f(x) = \frac{\pi \operatorname{cosec} \pi(\frac{1}{2}\beta-\frac{1}{2}\lambda)}{2^{\lambda+1}\Gamma(1+\frac{1}{2}\alpha+\frac{1}{2}\beta)}x^{\lambda+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(1+\frac{1}{2}\alpha+\frac{1}{2}\lambda+n)x^{2n}}{n!\,\Gamma(1+\lambda+\mu n)\Gamma(1+\frac{1}{2}\lambda-\frac{1}{2}\beta+n)}$$

$$+\frac{\pi \operatorname{cosec} \pi(\frac{1}{2}\lambda-\frac{1}{2}\beta)}{2^{\beta+1}\Gamma(1+\frac{1}{2}\alpha+\frac{1}{2}\beta)}x^{\beta+\frac{1}{2}}\sum_{n=0}^{\infty}\frac{\Gamma(1+\frac{1}{2}\alpha+\frac{1}{2}\beta+n)x^{2n}}{n!\,\Gamma(1+\frac{1}{2}\beta-\frac{1}{2}\lambda+n)\Gamma(1+\lambda-\frac{1}{2}\mu+\frac{1}{2}\mu\beta+\mu n)}.$$

In particular if $\mu = 1$ we get the Hankel pair

$$g(x) = \frac{x^{\beta+\frac{1}{2}}}{(1+x^2)^{\frac{1}{2}\alpha+\frac{1}{2}\beta+1}}$$

and

$$f(x) = \frac{\pi \operatorname{cosec} \pi(\frac{1}{2}\beta-\frac{1}{2}\lambda)}{2^{\lambda+1}\Gamma(1+\frac{1}{2}\alpha+\frac{1}{2}\beta)}x^{\lambda+\frac{1}{2}}{}_1f_2(1+\tfrac{1}{2}\alpha+\tfrac{1}{2}\lambda;\ 1+\lambda,\ 1+\tfrac{1}{2}\lambda-\tfrac{1}{2}\beta;\ x^2)$$

$$+\frac{\pi \operatorname{cosec} \pi(\frac{1}{2}\lambda-\frac{1}{2}\beta)}{2^{\beta+1}\Gamma(1+\frac{1}{2}\alpha+\frac{1}{2}\beta)}x^{\beta+\frac{1}{2}}{}_1f_2(1+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta;\ 1+\tfrac{1}{2}\beta-\tfrac{1}{2}\lambda,\ 1+\tfrac{1}{2}\lambda+\tfrac{1}{2}\beta;\ x^2)$$

Example 3. As another illustration let us take

$$\chi(s) = \frac{2^{\frac{1}{2}3\lambda-\frac{1}{2}}2^{\frac{1}{2}s}}{\sqrt{\pi}\,\Gamma(2\lambda+3)}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\lambda+\tfrac{3}{4})\Gamma(\tfrac{1}{2}\lambda-\tfrac{1}{2}s+\tfrac{5}{4})\Gamma(\tfrac{1}{2}s+\tfrac{1}{4}+\tfrac{1}{2}\lambda).$$

This gives (Titchmarsh, 1937, p. 262)

$$g(x) = x^{\lambda+\frac{1}{2}}e^{-\frac{1}{4}x^2}D_{-2\lambda-3}(x)$$

and

$$f(x) = \frac{1}{2^{\lambda-\frac{1}{2}}}\frac{\sqrt{\pi}\,x^{\lambda+\frac{1}{2}}}{\Gamma(2\lambda+3)}\sum_{r=0}^{\infty}\frac{\Gamma(r+2\lambda+2)(-)^r x^r}{\Gamma(r+1)\Gamma(\lambda+1+\frac{1}{2}\mu r)}.\quad [Re\,\lambda > -1].$$

5. Theorem 4. *If $f(x^2)$ and $g(x)$ are J_λ^μ-transforms and $f(p)$ is the Laplace transform of $\phi(x)$, then*

$$g(x) = \frac{1}{\mu}2^{1-2/\mu-2\lambda/\mu+\lambda}x^{-(2/\mu)(\frac{1}{2}3\mu-1+\lambda\overline{\frac{1}{2}\mu-1})}\int_0^{\infty}\varphi(z)G(z)dz$$

where,

$$G(z) = z^{-1/\mu-\lambda/\mu+\frac{1}{2}\lambda-3/4}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma[(r+\lambda+1/\mu)+\frac{3}{4}-\frac{1}{2}\lambda]}{r!\,\Gamma[(1+\lambda+r/\mu)]}\left(\frac{1}{4}\right)^{r/\mu}\left(\frac{x^2}{z}\right)^{r/\mu}$$

provided the integrals

$$\int_0^\infty \left| e^{-y^2 z}\phi(z) \right| dz$$

and

$$\int_0^\infty \phi(z) G(z)\, dz$$

exist and

$$0 < \mu \leq 1, \quad Re\left(\frac{3}{4} + \frac{1}{\mu} - \frac{\lambda}{\mu}\ \overline{\frac{\mu}{2} - 1}\right).$$

is positive and not an integer.

Proof: We have

$$g(x) = \frac{1}{\mu}\, 2^{2-2/\mu-2\lambda/\mu+\lambda} \int_0^\infty (xy)^{-(2/\mu)(\frac{1}{4}3\mu-1+\lambda\overline{\frac{1}{2}\mu-1})} J^{1/\mu}_{1/\mu+\lambda/\mu-1}\left[\left(\frac{x^2y^2}{4}\right)^{1/\mu}\right] f(y^2)\, dy$$

and

$$f(p) = p \int_0^\infty e^{-pz}\phi(z)\, dz.$$

Therefore,

$$g(x) = \frac{1}{\mu} 2^{2-(2/\mu)-(2\lambda/\mu)+\lambda} x^{-(2/\mu)(\frac{1}{4}3\mu-1+\lambda\overline{\frac{1}{2}\mu-1})}$$

$$\times \int_0^\infty y^{-(2/\mu)(\frac{1}{4}3\mu-1+\lambda\overline{\frac{1}{2}\mu-1})} J^{1/\mu}_{(1/\mu)+(\lambda/\mu)-1}\left[\left(\frac{x^2y^2}{4}\right)^{1/\mu}\right] \times y^2 \int_0^\infty e^{-y^2 z}\phi(z)\, dz.$$

Changing the order of integration, we get

$$g(x) = \frac{1}{\mu} 2^{2-(2/\mu)-(2\lambda/\mu)+\lambda} x^{-(2/\mu)(\frac{1}{4}3\mu-1+\lambda\overline{\frac{1}{2}\mu-1})} \int_0^\infty \phi(z)\, dz$$

$$\times \int_0^\infty y^{-(2/\mu)(\frac{1}{4}3\mu-1+\lambda\overline{\frac{1}{2}\mu-1})+2} e^{-y^2 z} J^{1/\mu}_{(1/\mu)+(\lambda/\mu)-1}\left[\left(\frac{x^2y^2}{4}\right)^{1/\mu}\right] dy$$

Expanding the Bessel function and integrating term by term, a process justified since the Bessel function involved is an integral function for $\mu > 0$, we get

$$g(x) = \frac{1}{\mu} 2^{1-(2/\mu)-(2\lambda/\mu)+\lambda} x^{-(2/\mu)(\frac{1}{4}3\mu-1+\lambda\overline{\frac{1}{2}\mu-1})} \int_0^\infty z^{-(1/\mu)-(\lambda/\mu)+\frac{1}{2}\lambda-3/4}\phi(z)$$

$$\times \sum_{r=0}^\infty \frac{(-)^r \Gamma[(\overline{r+\lambda+1}/\mu)+\frac{3}{4}-\frac{1}{2}\lambda](+\frac{1}{4})^{r/\mu}}{r!\,\Gamma(\overline{1+\lambda+r/\mu})} \left(\frac{x^2}{z}\right)^{r/\mu} dz$$

which proves our result.

In order to justify the change in the order of integration let us first study the behaviour of $G(z)$ for large and small values of z.

From the definition

$$G(z) = \int_0^\infty y^{-(2/\mu)(\frac{1}{4}3\mu-1+\lambda\overline{\frac{1}{2}\mu-1})} e^{-y^2 z} J^{1/\mu}_{(1/\mu)+(\lambda/\mu)-1}\left[\left(\frac{x^2y^2}{4}\right)^{1/\mu}\right] dy$$

which can be written after a slight change in the variable as

$$\frac{1}{2}\int_0^\infty y^{-(1/\mu)(\frac{1}{4}8\mu-1+\lambda\overline{\frac{1}{2}\mu-1})}e^{-yz}J^{1/\mu}_{(1/\mu)+(\lambda/\mu)-1}\left[\left(\frac{x^2}{4}y\right)^{1/\mu}\right]dy$$

If $(1/\mu+\frac{3}{4}-(\lambda/\mu)\overline{\frac{1}{2}\mu-1})$ is not an integer then the function

$$y^{-(1/\mu)(\frac{1}{4}8\mu-1+\lambda\overline{\frac{1}{2}\mu-1})}J^{1/\mu}_{(1/\mu)+(\lambda/\mu)-1}\left[\left(\frac{x^2}{4}y\right)^{1/\mu}\right]$$

satisfies all the conditions of the Watson's Lemma (10) regarding the asymptotic expansions for functions representable by the Laplace integral and is

$$O(y^{-(1/\mu)(\frac{1}{4}8\mu-1+\lambda\overline{\frac{1}{2}\mu-1})+\frac{1}{4}}),\quad \text{for small } y.$$

Hence from Watson's Lemma

$$G(z) = O(z^{(1/\mu)(\frac{1}{4}8\mu-1+\lambda\overline{\frac{1}{2}\mu-1})-8/2}),\quad \text{for large } z.$$

Also our definition of $G(z)$ shows that

$$G(z) = O(1),\quad \text{for small } z.$$

Now the inversion of the order of integration is justified assuming that

$$\int_0^\infty |\, e^{-y^2z}\phi(z)\,|\, dz$$

and

$$\int_0^\infty \phi(z)G(z)\,dz$$

are convergent and $0<\mu\leq 1$, $Re(\frac{3}{4}+(1/\mu)-(\lambda/\mu)\overline{\frac{1}{2}\mu-1})>0$.

Example 1: Consider the operational relation

$$x^{-\rho-1}e^{-1/x} \doteqdot 2p^{\frac{1}{2}\rho+1}k_\rho(2\sqrt{p}).$$

Hence,

$$f(x) = 2x^{\rho+2}k_\rho(2x)$$

which gives us

$$g(x) = 2^{9/2}x^{-3}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{5}{2}+2\rho-\mu\rho+\mu n)}{n\,!}\left(-\frac{4}{x^2}\right)^n$$

as the $J^{\mu}_{2\rho+8/2}$-transform of $f(x)$ valid under the conditions

$$Re(\tfrac{5}{2}+2\rho-\mu\rho)>0 \quad \text{atd} \quad Re\,\rho>-\tfrac{5}{4},\quad 0<\mu\leq 1.$$

Therefore, our theorem gives that

$$\sum_{n=0}^{\infty}\frac{\Gamma(\frac{5}{2}+2\rho-\mu\rho+\mu n)}{n\,!}\left(-\frac{4}{x^2}\right)^n$$

$$= \frac{1}{\mu}2^{2\rho-(4\rho/\mu)-(5/\mu)-2}x^{(5/\mu)+(4\rho/\mu)-2\rho}\int_0^\infty z^{-\rho-1}e^{-1/z}G(z)\,dz$$

valid under the conditions $Re(2\rho+5/2+\mu)>0$; $0<\mu\leq 1$.

An interesting; particular case of this integral is obtained by taking $\mu=\frac{1}{2}$. We get

$$\Gamma(5+3\rho)e^{2/x^4}D_{-5-3\rho}\left(\frac{2\sqrt{2}}{x^2}\right)$$

$$= 2^{-(10/2)-(9/2)\rho}x^{10+6\rho}\int_0^\infty z^{-4\rho-6}e^{-1/z}\times\sum_{r=0}^\infty \frac{\Gamma(2r+3\rho+5)}{r!\Gamma(2r+4\rho+5)}\left(-\frac{x^4}{16z^2}\right)^r dz$$

After an obvious change in the variable and simplification we get

$$x^{-10-6\rho}e^{2/x^4}D_{-5-3\rho}\left(\frac{2\sqrt{2}}{x^2}\right)$$

$$= \frac{1}{\Gamma(5+3\rho)}2^{-(10/2)-(11/2)\rho}\int_0^\infty e^{-y}y^{4\rho+4}{}_2f_2\left(\begin{matrix}\frac{3}{2}\rho+\frac{5}{2},\ \frac{3}{2}\rho+3\,;\\ 2\rho+\frac{5}{2},\ 2\rho+3\,;\end{matrix}\ -\frac{x^4y^2}{10}\right)dy$$

valid for $Re\rho>-5/4$, which is a particular case of a known operational relation due to Gupta (1948).

6. **Theorem 5.** *If $f(x)$ and $g(x)$ are J^μ_λ-transforms and $g(p)$ is the Laplace transform of $\psi(s)$ then,*

$$f(x) = \int_0^\infty \psi(s)G(s)ds$$

where,

$$G(s) = 2^{3/2}x^{-2}\sum_{n=0}^\infty \frac{\Gamma(\frac{1}{2}n+\frac{1}{2}\lambda+\frac{5}{4})(-2s/x)^n}{n!\Gamma(1+\lambda-\frac{1}{4}5\mu-\frac{1}{2}\mu\lambda-\frac{1}{2}\mu n)}$$

under the conditions $0<\mu<1$, and $Re(\lambda+\frac{5}{2})$ is positive and not an integer.

Proof: We have

$$f(x) = \left(\frac{1}{2}\right)^\lambda\int_0^\infty (xy)^{\lambda+\frac{1}{2}}J^\mu_\lambda\left(\frac{x^2y^2}{4}\right)g(y)dy$$

and

$$g(p) = p\int_0^\infty e^{-ps}\psi(s)ds.$$

Therefore

$$f(x) = \left(\frac{1}{2}\right)^\lambda\int_0^\infty (xy)^{\lambda+\frac{1}{2}}J^\mu_\lambda\left(\frac{x^2y^2}{4}\right)dy.y\int_0^\infty e^{-ys}\psi(s)ds.$$

Changing the order of integration, we get

$$f(x) = (\tfrac{1}{2})^\lambda x^{\lambda+\frac{1}{2}}\int_0^\infty \psi(s)ds\int_0^\infty y^{\lambda+3/2}J^\mu_\lambda\left(\frac{x^2y^2}{4}\right)e^{-ys}dy \tag{6.1}$$

Expanding e^{-ys} in ascending powers of y and integrating term by term the y-integral with the help of the integral (2.2*a*), we get

$$f(x) = 2^{3/2}x^{-2}\int_0^\infty \psi(s)\sum_{n=0}^\infty \frac{\Gamma(\frac{1}{2}n+\frac{1}{2}\lambda+\frac{5}{4})}{n!\Gamma(1+\lambda-\frac{1}{4}5\mu-\frac{1}{2}\mu\lambda-\frac{1}{2}\mu n)}\left(-\frac{2s}{x}\right)^n ds$$

The change in the order of integration is justified if

(i) $$\int_0^\infty y^{\lambda+3/2}J^\mu_\lambda\left(\frac{x^2y^2}{4}\right)e^{-ys}dy$$

and

(ii) $$\int_0^\infty e^{-ys}\psi(s)ds$$

are absolutely convergent and the repeated integral (6.1) exists.

Assuming that the integral (ii) is absolutely convergent, the integral (i) is absolutely convergent for $0<\mu<1$, $Re\,\lambda>-\frac{5}{2}$ and the repeated integral also exists for $0<\mu<1$ due to the behaviour of $G(s)$ as discussed below.

Hence, the theorem is true for $Re\,\lambda>-\frac{5}{2}$ and $0<\mu<1$.

The term by term integration is also valid due to the uniform convergence of the exponential series and because $G(s)$ is also uniformly convergent for $0<\mu<1$.

As regards the behaviour of $G(s)$ which is defined by

$$G(s) = (\tfrac{1}{2})^\lambda x^{\lambda+\frac{1}{2}}\int_0^\infty e^{-ys}J_\lambda^\mu\left(\frac{x^2y^2}{4}\right).y^{\lambda+3/2}dy$$

it follows that

$$G(s) = O(1) \text{ for small } s.$$

If $(\lambda+\frac{3}{2})$ is not an integer the function $y^{\lambda+3/2}J_\lambda^\mu(\frac{1}{4}x^2y^2)$ satisfies all the conditions of Watson's Lemma (Watson, 1944) for finding asymptotic expansions of functions representable by Laplace Integral and is

$$O(y^{\lambda+3/2}), \quad \text{for small } y.$$

Hence, from Watson's Lemma

$$G(s) = O(s^{-\lambda-5/2}), \quad \text{for large } s.$$

We shall now give some illustrations of the above theorem. This theorem is useful in evaluating different integrals involving the new function $G(s)$ defined above.

Example 1: Let us consider the operational relation (Hari Shankar, 1943)

$$p^{2k-\lambda}e^{\frac{1}{2}p^2}W_{-k,m}(p^2) \doteqdot \frac{x^\lambda}{\Gamma(\lambda+1)}\,{}_2F_2\left(\begin{matrix}k+\frac{1}{2}+m,\ k+\frac{1}{2}-m\\ \frac{1}{2}\lambda+\frac{1}{2},\ \frac{1}{2}\lambda+1;\end{matrix}\ \frac{-x^2}{4}\right)\ (Re\lambda>-1).$$

Taking,

$$g(p) = p^{2k-\lambda}e^{\frac{1}{2}p^2}W_{-k,m}(p^2)$$

we get

$$f(x) = (\tfrac{1}{2})^\lambda\int_0^\infty (xy)^{\lambda+\frac{1}{2}}J_\lambda^\mu\left(\frac{x^2y^2}{4}\right)\cdot y^{2k-\lambda}e^{\frac{1}{2}y^2}W_{-k,m}(y^2)dy$$

$$= (\tfrac{1}{2})^{\lambda+1}x^{\lambda+\frac{1}{2}}\int_0^\infty y^{k-\frac{1}{4}}J_\lambda^\mu\left(\frac{x^2}{4}\cdot y\right)e^{\frac{1}{2}y}W_{-k,m}(y)dy$$

Note: It may be remarked that the above theorem has been proved only for $0<\mu<1$. In order to extend the above result for the case $\mu=1$ *i.e.*, for the ordinary Hankel Transform we have to proceed as follows:

Putting $\mu=1$ and expanding the Bessel function in (6.1) in ascending powers of y and integrating term by term we get,

$$f(x) = \frac{2^{3/2}}{\sqrt{\pi}}x^{\lambda+\frac{1}{2}}\int_0^\infty s^{-\lambda-5/2}\psi(s)\,{}_2F_1(\tfrac{1}{2}\lambda+\tfrac{5}{4},\ \tfrac{1}{2}\lambda+\tfrac{7}{4}: 1+\lambda;\ -x^2/s^2)ds. \quad (i)$$

Substituting the integral (Dhar, 1936)

$$x^{k}W_{-k,m}(x)e^{\frac{1}{2}x} = \frac{1}{\Gamma(\frac{1}{2}+k-m)\Gamma(\frac{1}{2}+k+m)} \times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(s)\Gamma(k+m+\tfrac{1}{2}-s)\Gamma(k-m+\tfrac{1}{2}-s)x^{s}ds$$

and changing the order of integration we get

$$f(x) = (\tfrac{1}{2})^{\lambda+1}\frac{x^{\lambda+\frac{1}{2}}}{\Gamma(\frac{1}{2}+k-m)\Gamma(\frac{1}{2}+k+m)}\cdot\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(s)\Gamma(k+m+\tfrac{1}{2}-s)\Gamma(k-m+\tfrac{1}{2}-s)ds \times \int_{0}^{\infty} y^{s-\frac{1}{2}}J_{\lambda}^{\mu}\left(\frac{x^{4}}{4}y\right)dy$$

Integrating the y-integral with the help of the integral (2.2a) and evaluating the residues at the poles of the integrand, we get

$$f(x) = (\tfrac{1}{2})^{\lambda+1}\frac{x^{\lambda+\frac{1}{2}}}{\Gamma(\frac{1}{2}+k-m)\Gamma(\frac{1}{2}+k+m)}\left\{\left(\frac{x^{2}}{4}\right)^{-3/4}\sum_{r=0}^{\infty}\frac{\Gamma(-r+\frac{3}{4})\Gamma(k+\frac{1}{2}+m+r)\Gamma(k+\frac{1}{2}-m+r)}{\Gamma(1+\lambda+\mu r-\frac{3}{4})r!(\frac{1}{4}-x^{2})^{r}} + \sum_{r=0}^{\infty}\frac{\Gamma(-r-\frac{3}{4})\Gamma(k+\frac{5}{4}+m+r)\Gamma(k+\frac{5}{4}-m+r)}{\Gamma(1+\lambda+\mu r)r!}\left(\frac{-x^{2}}{4}\right)^{r}\right.$$

valid for $0<\mu\leq 1$, $Re(k\pm m+\frac{3}{4})>0$ and $Re\lambda>-1$. The change in the order of integration is easily justifiable. Our theorem therefore gives that

$$\frac{(\frac{1}{2})^{\lambda+1}x^{\lambda+\frac{1}{2}}}{\Gamma(k+\frac{1}{2}-m)\Gamma(k+\frac{1}{2}+m)}\left\{\left(\frac{x^{2}}{4}\right)^{-3/4}\sum_{r=0}^{\infty}\frac{\Gamma(-r+\frac{3}{4})\Gamma(k+\frac{1}{2}+m+r)\Gamma(k+\frac{1}{2}-m+r)}{\Gamma(1+\lambda+\mu r-\frac{3}{4})r!}\left(\frac{-x^{2}}{4}\right)^{r} + \sum_{r=0}^{\infty}\frac{\Gamma(-r-\frac{3}{4})\Gamma(k+\frac{5}{4}+m+r)\Gamma(k+\frac{5}{4}-m+r)}{r!\Gamma(1+\lambda+\mu r)}\left(\frac{-x^{2}}{4}\right)^{r}\right\}$$

$$= \frac{1}{\Gamma(\lambda+1)}\int_{0}^{\infty}s^{\lambda}{}_{2}F_{2}\left(\begin{matrix}k+\frac{1}{2}+m, & k+\frac{1}{2}-m\,; \\ \frac{1}{2}\lambda+\frac{1}{2}, & \frac{1}{2}\lambda+1\,;\end{matrix}\; -\frac{s^{2}}{4}\right)G(s)ds$$

valid for $Re\lambda>-1$, $Re(k\pm m+\frac{7}{4})>0$.

Example 2 · Next, let us consider the Operational relation

$$\frac{p^{2}}{p^{2}+b^{2}} \doteqdot \cos bx$$

then

$$g(p) = \frac{p^{2}}{p^{2}+b^{2}}$$

and consequently

$$f(x) = (\tfrac{1}{2})^{\lambda}x^{\lambda+\frac{1}{2}}\int_{0}^{\infty}y^{\lambda+\frac{1}{2}}J_{\lambda}^{\mu}\left(\frac{x^{2}y^{2}}{4}\right)\cdot\frac{y^{2}}{y^{2}+b^{2}}dy$$

Substituting the integral of Barnes type for the J_{λ}^{μ}-function and changing the order of integration, which is justified due to the absolute convergence of the integrals involved, we get

$$f(x) = (\tfrac{1}{2})^{\lambda}x^{\lambda+\frac{1}{2}}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(-s)}{\Gamma(1+\lambda+\mu s)}\left(\frac{x^2}{4}\right)^s ds\int_0^{\infty}\frac{y^{\lambda+2s+5/2}}{y^2+b^2}\,dy$$

$$= (\tfrac{1}{2})^{\lambda+1}x^{\lambda+\frac{1}{2}}b^{\lambda+3/2}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(-s)\Gamma(-\frac{1}{2}\lambda-\frac{3}{4}-s)\Gamma(\frac{1}{2}\lambda+s+\frac{7}{4})}{\Gamma(1+\lambda+\mu s)}\left(\frac{x^2b^2}{4}\right)^s ds$$

Evaluating the integral by the Calculus of Residues, we get

$$f(x) = (\tfrac{1}{2})^{\lambda+1}x^{\lambda+\frac{1}{2}}b^{\lambda+3/2}\Bigg[\sum_{n=0}^{\infty}\frac{\Gamma(-\frac{1}{2}\lambda-\frac{3}{4}-n)\Gamma(\frac{1}{2}\lambda+n+\frac{7}{4})}{\Gamma(1+\lambda+\mu n)\,n\,!}\left(-\frac{x^2b^2}{4}\right)^n$$

$$+\left(\frac{x^2b^2}{4}\right)^{-\frac{1}{2}\lambda-3/4}\sum_{n=0}^{\infty}\frac{\Gamma(-n+\frac{1}{2}\lambda+\frac{3}{4})\Gamma(n+1)}{\Gamma(1+\lambda+\mu n-\frac{1}{2}\mu\lambda-\frac{1}{4}3\mu)}\frac{1}{n\,!}\left(-\frac{x^2b^2}{4}\right)^n\Bigg]$$

valid for $0<\mu\leq 1$, $Re\ \lambda>-\frac{3}{2}$.

Our theorem gives

$$f(x) = \int_0^{\infty} G(s)\cos bs\,ds$$

valid under the conditions stated in the theorem.

7. **Theorem 6,** *If $x^{\lambda-11/6}f(x)$ is Humbert transform of $g(x)$ where $f(x)$ satisfies the condition.*

(1) $f(x)=0, \quad x>1$

(2) $f(x)$ is not a function of λ,

where λ is an integer, say n, $(n = 0, 1, 2, 3\ldots)$, then

$$\frac{1}{\sqrt{2}}e^{2s^4}\sum_{n=0}^{\infty}\frac{(-)^n x^{n-\frac{1}{4}}D_{2n+1}(2\sqrt{2}s^2)g(x)}{(2n+1)\,!} = \int_0^1 y^{-4/3}\,bei\,(2xys)\,f(y)dy \tag{7.1}$$

for all finite values of s within the circle $|xy| = 2\sqrt{2}$ and provided the above equation has a meaning.

Proof: Since

$$f(x)=0, \quad x>1$$

we have from (1.2) for $\mu=\frac{1}{2}$

$$g(x) = 2^{-1-\lambda}\sqrt{\pi}\int_0^1 (xy)^{(\lambda/3)+(11/6)}J_{\lambda,\,\lambda+\frac{1}{2}}(\tfrac{3}{4}x^{4/3}y^{4/3})y^{\lambda-(11/6)}f(y)dy$$

Putting $\lambda = n$, we get

$$x^{n-(11/6)}\frac{2}{\sqrt{\pi}}g(x) = \frac{1}{2^n}\int_0^1 (xy)^{4/3n}J_{n,n+\frac{1}{2}}(\tfrac{3}{4}x^{4/3}y^{4/3})f(y)dy.$$

Multiplying both sides by $(-)^nD_{2n+1}(2\sqrt{2}s^2)/(2n+1)\,!$ and summing for n, from zero to infinity, we get

$$\frac{2}{\sqrt{\pi}}x^{-11/6}\sum_{n=0}^{\infty}\frac{(-)^n x^n D_{2n+1}(2\sqrt{2}s^2)g(x)}{(2n+1)\,!}$$

$$=\sum_{n=0}^{\infty}\frac{(-)^nD_{2n+1}(2\sqrt{2}s^2)}{2^n(2n+1)\,!}\int_0^1 (xy)^{4/3n}J_{n,n+\frac{1}{2}}(\tfrac{3}{4}x^{4/3}y^{4/3})f(y)dy$$

If the change in the order of integration and summation be permissible, we have

$$\sqrt{(4/\pi)}x^{-(11/6)} \sum_{n=0}^{\infty} \frac{(-x)^n D_{2n+1}(2\sqrt{2}z^2)g(x)}{(2n+1)!}$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-)^n}{2^n(2n+1)!} D_{2n+1}(2\sqrt{2}z^2)(xy)^{4/8n} J_{n,n+\frac{1}{2}}(\tfrac{3}{4}x^{4/3}y^{4/3})f(y)dy$$

Now, summing the series by the help of a result due to Varma (1941), we get

$$\frac{1}{\sqrt{2}} e^{2z^4} \sum_{n=0}^{\infty} \frac{(-)^n x^{n-\frac{1}{2}} D_{2n+1}(2\sqrt{2}z^2)g(x)}{(2n+1)!} = \int_0^1 y^{-4/3} bei(2xyz)f(y)dy.$$

The change in the order of integration and summation is permissible since the series is uniformly convergent for all finite values of z within the circle $|xy| = 2\sqrt{2}$, as shown by Varma (1941).

Hence, the result is true provided, in addition to the above conditions the series and the integral in (7.1) converge.

As an illustration let us take

$$\left.\begin{aligned} f(x) &= x^{7/8} \qquad & x<1 \\ &= 0 \qquad & x>1 \end{aligned}\right\}$$

Then

$$g(x) = \frac{\sqrt{\pi}}{2^{n+1}} z^{n/8+\frac{1}{2}} J_{n+1,\, n+\frac{1}{2}}(\tfrac{3}{4}x^{4/3})$$

Therefore our theorem gives that

$$\frac{\sqrt{\pi}}{2\sqrt{2}} e^{2z^4} \sum_{n=0}^{\infty} \frac{(-)^n x^{4/8n}}{2^n(2n+1)!} D_{2n+1}(2\sqrt{2}z^2) J_{n+\frac{1}{2},\, n+1}(\tfrac{3}{4}x^{4/3}) = \int_0^1 y \operatorname{bei}(2xyz)dy.$$

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STATIONARY GRAVITATIONAL FIELDS

By

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Abstract

For a stationary gravitational field as defined here no non-trivial infinitesimal variations of the metric potentials exist if the material energy tensor is not varied. The additional equations providing a criterion for the existence of a stationary field are derived. Three particular metrics are examined and one of the results obtained is that Schwarzschild's metric of external solution is stationary so far as variations of a certain type preserving the spherical symmetry are concerned. Stationary gravitational metrics have a special significance in relation to the molar character of general relativity as a field theory.

1. Introduction

The present investigation starts from a consideration of the well-known Poisson equation in the Newtonian theory of gravitation, *viz.*,

$$\nabla^2 V = -4\pi\rho \tag{1}$$

If we consider a small variation in the potential function V, say ϵV_1, where ϵ is a small parameter the corresponding equation becomes

$$\nabla^2(V+\epsilon V_1) = -4\pi\rho - \epsilon 4\pi\rho_1. \tag{2}$$

It is obvious that $\rho_1 = 0$ implies $\nabla^2 V_1 = 0$. In the general theory of relativity we have equations,

$$T_{\mu\nu} = G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} G, \tag{3}$$

which are not linear like Poisson's equation If the metrical potentials $g_{\mu\nu}$ undergo a similar small change and become $g_{\mu\nu}+\epsilon\varphi_{\mu\nu}$ the material energy tensor changes from $T_{\mu\nu}$ to $T_{\mu\nu}+\epsilon T'_{\mu\nu}$ The question arises, 'What does $T'_{\mu\nu}=0$ imply?'. It is found that $T'_{\mu\nu}=0$ does not necessarily mean $\varphi_{\mu\nu}=0$ but it imposes certain restrictions upon the original metrical potentials which we investigate in the following. For the sake of definiteness let us consider first the case when $T_{\mu\nu}=0$. The field equations,

$$G_{\mu\nu} = 0, \tag{4}$$

which are satisfied by $g_{\mu\nu}$ can as well be satisfied by $g_{\mu\nu}+\epsilon\varphi_{\mu\nu}$ if $\varphi_{\mu\nu}$ satisfy certain obvious conditions. Thus if certain parameters $m_1, m_2 \ldots$ etc. appear in $g_{\mu\nu}$ we can have

$$g_{\mu\nu} + \frac{\partial g_{\mu\nu}}{\partial m_1}\delta m_1 + \frac{\partial g_{\mu\nu}}{\partial m_2}\delta m_2 + \cdots \tag{5}$$

also as a solution, $\delta m_1, \delta m_2, \ldots$ etc. being of the order of ϵ. For example, if

$$ds^2 = -(1-2m/r)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 + (1-2m/r)dt^2 \tag{6}$$

is a solution we can as well have another solution

$$ds^2 = -dr^2\left[(1-2m/r)^{-1}+(1-2m/r)^{-2}2\frac{\delta m}{r}\right]-r^2d\theta^2-r^2\sin^2\theta d\varphi^2$$
$$+(1-2m/r-2\delta m/r)dt^2 \qquad (7)$$

It is also true that if we have a change of coordinates as defined by

$$x'^i = x^i + f^i(x^a), \qquad (8)$$

where f^i are infinitesimal functions,

$$g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} = g_{\alpha\beta}(\delta^\alpha_\mu + f^\alpha_{,\mu})(\delta^\beta_\nu + f^\beta_{,\nu})$$

so that

$$g'_{\mu\nu} = g_{\mu\nu} + g_{\alpha\nu}f^\alpha_{,\mu} + g_{\mu\alpha}f^\alpha_{,\nu} \qquad (9)$$

to the first order where $f^i_{,a}$ stands for $\partial f^i/\partial x^a$. We have thus a solution with a small change in $g_{\mu\nu}$ which cannot be considered to be a new solution In the general case we expect that if $g_{\mu\nu}+\epsilon\varphi_{\mu\nu}$ also satisfy the field equations (4) we must have

$$\frac{\partial G_{\mu\nu}}{\partial g_{\alpha\beta}}\varphi_{\alpha\beta} + \frac{\partial G_{\mu\nu}}{\partial g_{\alpha\beta,\gamma}}\varphi_{\alpha\beta,\gamma} + \frac{\partial G_{\mu\nu}}{\partial g_{\alpha\beta,\gamma\delta}}\varphi_{\alpha\beta,\gamma\delta} = 0, \qquad (10)$$

where $\varphi_{\alpha\beta,\gamma}$ stands for $\partial\varphi_{\alpha\beta}/\partial x^\gamma$ etc. Here are ten equations for $\varphi_{\alpha\beta}$. As we have seen above it is clear that if $\varphi_{\mu\nu}$ is of the form

$$\epsilon\varphi_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial m_1}\delta m_1 + \frac{\partial g_{\mu\nu}}{\partial m_2}\delta m_2 + \cdots + g_{\alpha\nu}f^\alpha_{,\mu} + g_{\alpha\mu}f^\alpha_{,\nu} + \cdots \qquad (11)$$

the equations are integrable The variations considered in (11) are treated as trivial. When there is no solution of equations (10) giving a non-trivial $\varphi_{\mu\nu}$ the metrical potentials $g_{\mu\nu}$ define a stationary field The search for stationary solutions is in a way a search for isolated solutions in the neighbourhood of which a non-trivial physically distinct solution does not exist. A neighbouring solution involving a slight variation of parameter such as the mass parameter is to be treated as trivial Nor do we consider as physically distinct a solution which is in the neighbourhood of a given solution merely by virtue of an infinitesimal transformation

When the field equations are

$$G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G = \kappa T_{\mu\nu} \qquad (12)$$

trivial variations as given by (11) will cause a change $\epsilon T'_{\mu\nu}$ in the material energy tensor which we may denote by $\delta T_{\mu\nu}$. We wish to find out if any non-trivial variations of the metrical potentials exist for which $\delta T_{\mu\nu} = 0$. The solutions of the field equations will now be further restricted by additional conditions and it will be of interest to test if any of the well-known line-elements satisfy the conditions for being stationary.

Eddington (1920) has emphasized certain aspects of general relativity as a molar field theory. He refers to the macroscopic field equations (12) as "obtained solely from the law of gravitation by the process of averaging." Each metric potential must be treated as having a statistical fringe $\pm\delta g_{\mu\nu}$. Only a microscopic theory can relate $\delta g_{\mu\nu}$

to the statistical fluctuations of matter and energy at (x, y, z, t). In the absence of any such theory the metrics for which non-trivial values of $\delta g_{\mu\nu}$, corresponding to $\delta T_{\mu\nu} = 0$, do not exist have a special physical significance.

2. The Additional Conditions

Consider the line-element,

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu. \tag{13}$$

In empty space we have

$$H_{\mu\nu} = 0 \tag{14}$$

or otherwise

$$T_{\mu\nu} = H_{\mu\nu} - \tfrac{1}{2} h_{\mu\nu} H, \tag{15}$$

where $H_{\mu\nu}$ is the contracted Riemann-Christoffel curvature tensor corresponding to the metric space given by (13). We now consiber infinitesimal variations of the metrical potentials so that $h_{\mu\nu}$ is replaced by $g_{\mu\nu}$ where

$$g_{\mu\nu} = h_{\mu\nu} + \epsilon \varphi_{\mu\nu}. \tag{16}$$

$\varphi_{\mu\nu}$ is obviously a tensor. Thus we have a new field which can be considered to be a disturbed form of the original gravitation field. The distribution of matter will accordingly vary and it can be expressed as

$$T_{\mu\nu} + \delta T_{\mu\nu} = G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} G, \tag{17}$$

where $G_{\mu\nu}$ is now the corresponding contracted curvature tensor for the metric (16). We have

$$g = h + \epsilon \varphi_{\mu\nu} \frac{\partial h}{\partial h_{\mu\nu}}, \tag{18}$$

where g and h are the determinants of the metrics $g_{\mu\nu}$ and $h_{\mu\nu}$ respectively. We get

$$g^{\alpha\beta} = h^{\alpha\beta} + \epsilon B^{\alpha\beta}, \tag{19}$$

where

$$B^{\alpha\beta} = \varphi_{pq} \frac{1}{h} \frac{\partial^2 h}{\partial h_{\alpha\beta} \partial h_{pq}} - h^{\alpha\beta} \varphi_{\mu\nu} h^{\mu\nu} \tag{20}$$

and p, q run over only three values out of 1, 2, 3, 4 such that $p \neq \alpha$, $q \neq \beta$. Herein and in what follows terms only up to the first order of small quantities are retained neglecting ϵ^2 and higher powers.

Also we know that

$$\left\{ {\lambda \atop \mu\nu} \right\} = g^{\rho\lambda} [\mu\nu, \rho] \tag{21}$$

and

$$[\mu\nu, \rho] = \tfrac{1}{2} \left(\frac{\partial g_{\mu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right). \tag{22}$$

Substituting in the above the values of $g_{\mu\nu}$ from (16) we have

$$\left\{ {\lambda \atop \mu\nu} \right\} = \Gamma^\lambda_{\mu\nu} + \epsilon K^\lambda_{\mu\nu}, \tag{23}$$

where

$$K^\lambda_{\mu\nu} = B^{\lambda\rho} \Gamma_{\mu\nu,\rho} + h^{\lambda\rho} \Phi_{\mu\nu,\rho}. \tag{24}$$

$\Gamma^{\lambda}_{\mu\nu}$ and $\Gamma_{\mu\nu,\lambda}$ being the Christoffel three-index symbols corsesponding to the metric $h_{\mu\nu}$ and $\Phi^{\lambda}_{\mu\nu}$ and $\Phi_{\mu\nu,\lambda}$ being three-index symbols similarly constructed in terms of $\varphi_{\mu\nu}$. From (23) it is obvious that $K^{\lambda}_{\mu\nu}$ is a tensor.

Now
$$G_{\mu\nu} = \begin{Bmatrix} \alpha \\ \sigma\mu \end{Bmatrix}\begin{Bmatrix} \sigma \\ \alpha\nu \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}\begin{Bmatrix} \sigma \\ \alpha\sigma \end{Bmatrix} + \frac{\partial}{\partial x^{\nu}}\begin{Bmatrix} \sigma \\ \mu\sigma \end{Bmatrix} - \frac{\partial}{\partial x^{\sigma}}\begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix} \tag{25}$$

which on substitution from (23) gives

$$G_{\mu\nu} = H_{\mu\nu} + \epsilon[K^{\sigma}_{\mu\sigma;\nu} - K^{\sigma}_{\mu\nu;\sigma}], \tag{26}$$

where a semi-colon (;) preceding a suffix denotes a covariant differentiation for the metric $h_{\mu\nu}$. Again using (26) together with (17) we get

$$\delta T_{\mu\nu} = \epsilon[K^{\sigma}_{\mu\sigma;\nu} - K^{\sigma}_{\mu\nu;\sigma} - \tfrac{1}{2}h_{\mu\nu}h^{\alpha\beta}(K^{\sigma}_{\alpha\sigma;\beta} - K^{\sigma}_{\alpha\beta,\sigma}) - \tfrac{1}{2}H_{\alpha\beta}(h^{\alpha\beta}\varphi_{\mu\nu} + h_{\mu\nu}B^{\alpha\beta})]. \tag{27}$$

We say that $h_{\mu\nu}$ defines a stationary gravitational field if there exists no non-trivial solution $\varphi_{\mu\nu}$ of the equations,

$$\delta T_{\mu\nu} = 0,$$

that is,

$$K^{\sigma}_{\mu\sigma;\nu} - K^{\sigma}_{\mu\nu,\sigma} - \tfrac{1}{2}h_{\mu\nu}h^{\alpha\beta}(K^{\sigma}_{\alpha\sigma,\beta} - K^{\sigma}_{\alpha\sigma;\beta} - K^{\sigma}_{\alpha\beta;\sigma}) - \tfrac{1}{2}H_{\alpha\beta}(h^{\alpha\beta}\varphi_{\mu\nu} + h_{\mu\nu}B^{\alpha\beta}) = 0. \tag{28}$$

In the case when $T_{\mu\nu} = 0$, the conditions (28) reduce to

$$K^{\sigma}_{\mu\sigma;\nu} - K^{\sigma}_{\mu\nu;\sigma} = 0, \tag{25}$$

as is obvious from (26).

3. Special Metrics

In the following we examine three well-known line-elements of physical interest from the point of view of their being stationary when the metrical potentials are varied in a restricted manner.

(*a*) The line-element of spherical symmetry in the static case,

$$ds^2 = -e^{\varphi}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi_0^2 + e^{\psi}dt^2, \tag{10}$$

where

$$\varphi = \varphi(r) \quad \text{and} \quad \psi = \psi(r)$$

The values of the material energy tensor (Tolman 1934*a*) are as follows:

$$\left.\begin{aligned} 8\pi T_1^{\,1} &= -e^{-\varphi}(\psi'/r + 1/r^2) + 1/r^2, \\ 8\pi T_2^{\,2} = 8\pi T_3^{\,3} &= -e^{-\varphi}\left(\frac{\psi''}{2} - \frac{\psi'\varphi'}{4} + \frac{\psi'^2}{4} + \frac{\psi' - \varphi'}{2r}\right), \\ 8\pi T_4^{\,4} &= e^{-\varphi}(\varphi'/r - 1/r^2) + 1/r^2. \end{aligned}\right\} \tag{31}$$

Now we consider infinitesimal variations of the metrical potentials such that the spherical symmetry of (30) is preserved. Thus we take

$$\bar{\varphi} = \varphi + \epsilon\varphi_1 \quad \text{and} \quad \bar{\psi} = \psi + \epsilon\psi_1,$$

where φ_1 and ψ_1 are functions of r alone.

Setting δT_1^1, δT_2^2 and δT_4^4 equal to zero we get the following three equations:

$$e^{-\varphi}\left(\frac{\psi'}{r}\varphi_1+\frac{\varphi_1}{r^2}-\frac{\psi_1'}{r}\right)=0, \tag{32}$$

$$-e^{-\varphi}\left(\frac{\psi_1''}{2}-\frac{\varphi_1'\psi'+\varphi'\psi_1'}{4}+\frac{\psi'\psi_1'}{2}+\frac{\psi_1'-\varphi_1'}{2r}\right)+e^{-\varphi}\varphi_1\left(\frac{\psi''}{2}-\frac{\varphi'\psi'}{4}+\frac{\psi'^2}{4}+\frac{\psi'-\varphi'}{2r}\right)=0, \tag{33}$$

and

$$e^{-\varphi}\frac{\varphi_1'}{r}-e^{-\varphi}\varphi_1\left(\frac{\varphi'}{r}-\frac{1}{r^2}\right)=0. \tag{34}$$

Equation (34) gives

$$\varphi_1=e^{\varphi}/r, \tag{35}$$

which together with (32) leads to

$$\psi_1'=(\psi'+1/r)e^{\varphi}/r. \tag{36}$$

Substituting in (33) from (35) and (36) we get

$$(\varphi'+\psi')(\psi'+1/r)=0, \tag{37}$$

which gives

$$\varphi'=-\psi' \tag{38}$$

or

$$\psi'=-1/r. \tag{39}$$

Relation (39) is physically untenable as it gives

$$-8\pi p=1/r^2 \tag{40}$$

from the first of the equations (31). The solution (38) gives Schwarzschild's line-element for which

$$e^{\varphi}=(1-2m/r)^{-1} \tag{41}$$

$$e^{\psi}=(1-2m/r). \tag{42}$$

Hence the only static line-element of the form (30) which is stationary is Schwarzschild's external solution when the variations of the metric potentials are such that the spherical symmetry is preserved.

(*b*) The non-static line-element of spherical symmetry,

$$ds^2=-e^{\varphi}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi_0^2+e^{\psi}dt^2, \tag{43}$$

where

$$\varphi=\varphi(r,t) \quad \text{and} \quad \psi=\psi(r,t).$$

The values of T_μ^ν (Tolman 1934*b*) are as follows:

$$\left.\begin{aligned}
8\pi T_1^1 &= -e^{-\varphi}\left(\frac{\psi'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2},\\
8\pi T_2^2 &= 8\pi T_3^3 = -e^{-\varphi}\left(\frac{\psi''}{2}-\frac{\varphi'\psi'}{4}+\frac{\psi'^2}{4}+\frac{\psi'-\varphi'}{2r}\right)+e^{\psi}\left(\frac{\ddot\varphi}{2}+\frac{\dot\varphi^2}{4}-\frac{\dot\varphi\dot\psi}{4}\right),\\
T_4^4 &= e^{-\varphi}\left(\frac{\varphi'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2},\\
T_1^4 &= e^{-\psi}.\dot\varphi/r,\\
8\pi T_4^1 &= -e^{-\varphi}.\dot\varphi/r.
\end{aligned}\right\} \tag{44}$$

Considering variations of φ and ψ similar to those in the above case (a) we put

$$\bar{\varphi} = \varphi + \epsilon\varphi_1 \quad \text{and} \quad \bar{\psi} = \psi + \epsilon\psi_1,$$

φ_1 and ψ_1 being functions of r and t. Equating to zero the first order variations in $T_\mu{}^\nu$ we find that the following equations arise:

$$\psi_1 = \varphi_1 = e^\varphi / r = re^\psi, \tag{45}$$

provided $\dot{\varphi} \not\equiv 0$. The equation arising out of $\delta T_2{}^2 = 0$ is

$$-e^{-\varphi}\left[\frac{\psi_1''}{2} - \frac{\varphi_1'\psi' + \varphi'\psi_1'}{4} + \frac{\psi'\psi_1'}{2} + \frac{\psi_1' - \varphi_1'}{2r}\right] + e^{-\varphi}\varphi_1\left[\frac{\psi''}{2} - \frac{\varphi'\psi'}{4} + \frac{\psi'^2}{4} + \frac{\psi' - \varphi'}{2r}\right] \tag{46}$$

$$+ e^{-\psi}\left[\frac{\ddot{\varphi}_1}{2} + \frac{\dot{\varphi}\dot{\varphi}_1}{2} - \frac{\dot{\varphi}_1\dot{\psi} + \dot{\varphi}\dot{\psi}_1}{4}\right] - e^{-\psi}\psi_1\left[\frac{\ddot{\varphi}}{2} + \frac{\dot{\varphi}^2}{4} - \frac{\dot{\varphi}\dot{\psi}}{4}\right] = 0$$

Substituting from (45) in (46) we get

$$\psi' \pm \dot{\psi}r + 1/r = 0, \tag{47}$$

which leads to the solution,

$$e^\varphi = rf(\tfrac{1}{2}r^2 \pm t), \quad e^\psi = 1/r.f(\tfrac{1}{2}r^2 \pm t). \tag{48}$$

corresponding to (48) the values of the material energy tensor are given by the following:

$$\left.\begin{aligned} 8\pi T_1{}^1 &= -f_1/rf^2 + 1/r^2, \\ 8\pi T_2{}^2 &= 8\pi T_3{}^3 = 0, \\ 8\pi T_4{}^4 &= f_1/rf^2 + 1/r^2, \\ 8\pi T_1{}^4 &= \pm f_1/f^2, \\ 8\pi T_4{}^1 &= \mp f_1/r^2 f^2, \end{aligned}\right\} \tag{49}$$

where f_1, denotes differential coefficient of the function f with respect to $(\frac{1}{2}r^2 \pm t)$. There is an internal inconsistency of the values given by (49) since the identity that must be satisfied in this case, by the material energy tensor, *viz*,

$$(T_1{}^1 - T_2{}^2)(T_4{}^4 - T_2{}^2) = T_1{}^4 T_4{}^1 \tag{50}$$

is not satisfied. Hence no non-static spherically symmetrical line-element stationary in the sense of this paper exists for variations consistent with spherical symmetry.

(c) The cosmological line-element,

$$ds^2 = dt^2 - \frac{[\varphi(t)]^2}{(1 + ar^2/4)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi_0{}^2), \tag{51}$$

where

$$a = 0 \quad \text{or} \quad \pm 1.$$

the values of $T_\mu{}^\nu$ are as follows (Tolman 1934c):

$$\left.\begin{aligned} 8\pi T_1{}^1 &= 8\pi T_2{}^2 = 8\pi T_3{}^3 = (a + 2\varphi\ddot{\varphi} + \dot{\varphi}^2)/\varphi^2 \\ \text{and} \quad 8\pi T_4{}^4 &= 3(a + \dot{\varphi}^2)/\varphi^2. \end{aligned}\right\} \tag{52}$$

We replace $\varphi(t)$ by $\varphi(t) + \epsilon.\varphi_1(t)$ so as to retain the isotropic form of (51). The two equations that arise on equating to zero the first order variations of $T_\mu{}^\nu$ are

$$\varphi^2\ddot{\varphi}_1-\varphi\varphi_1\ddot{\varphi}+(\varphi\dot{\varphi}\dot{\varphi}_1-a\varphi_1-\varphi_1\dot{\varphi}^2) = 0 \tag{53}$$

and

$$\varphi\dot{\varphi}\dot{\varphi}_1-a\dot{\varphi}_1-\varphi_1\dot{\varphi}^2 = 0,$$

which on eliminating φ_1, between them give

$$\varphi\dot{\varphi}-\dot{\varphi}^2 = a \tag{54}$$

for determining φ. As a first integral we get

$$\dot{\varphi}^2+a = \alpha\varphi^2, \tag{55}$$

where α is a constant of integration. But since

$$8\pi T_4^{\ 4} = 3(a+\dot{\varphi}^2)/\varphi^2 = 3\alpha \tag{56}$$

α must be positive or zero. If α is a non-zero positive constant we have from (52)

$$8\pi T_1^{\ 1} = -8\pi p = 3\alpha \tag{57}$$

which is physically untenable. If we put $\alpha = 0$ (55) gives

$$\dot{\varphi}^2+a = 0 \tag{58}$$

which implies that a must be negative and equal to -1. (58) then leads to

$$\varphi = (k \pm t), \tag{59}$$

k being a constant of integration. This gives the components of the material energy tensor as

$$\left.\begin{aligned} 8\pi T_1^{\ 1} = 8\pi T_2^{\ 2} = 8\pi T_3^{\ 3} = -8\pi p = 0 \\ \text{and} \qquad 8\pi T_4^{\ 4} = 8\pi\rho = 0 \end{aligned}\right\} \tag{60}$$

It has been verified that (59) gives flat space-time as is also expected from equations (60). Thus $a = +1$ is untenable, $a = -1$ leads to flat space-time and $a = 0$ is obviously a trivial case as can be seen from the equations (58) and (51).

We have neglected the cosmological constant λ in the above analysis. But even if λ is treated as non-zero it can be verified that no cosmological model of the type (51) exists which can be cosidered to be stationary for small variations of $\varphi(t)$ consistent with the isotropic form of the line-element.

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ON A THEOREM OF STONE-SAMUEL

By

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1. *Fundamental concepts.*

The object of this note is to give a simple proof of a theorem of Stone generalized by P. Samuel (1948, Theorem VII).

In this section, we shall review briefly the definitions and the results which are needed in our discussion.

Let L be a partially ordered set.* We say that L is a semi-lattice if

(1) any two given elements a, b of L have greatest lower bound $a \cap b$.

(2) L has a minimal element $O : O \leq a$ for all $a \in L$.

In semi-lattice L, the operation $\cap$ has the following properties:

$$a \cap a = a, \quad a \cap b = b \cap a, \quad a \cap (b \cap c) = (a \cap b) \cap c \quad \text{and} \quad a \cap O = O.$$

Any lattice with O is obviously a semi-lattice. Throughout this paper, we use the terminology '*lattice*' as a lattice with the element O.

A subset F of a semi-lattice L is called a *filter* if

(1) $O \bar{\in} F$,

(2) $F \ni a, b$ implies $F \ni a \cap b$,

(3) $F \ni a$, $a \leq x$ implies $x \in F$.

A filter U is called a *ultrafilter* or a maximal filter, if there is no filter containing properly U.

Then it is well known that there exists at least one ultrafilter containing a given filter.

If L is a lattice, we can consider a filter called prime. A filter P in a lattice is called *prime* if $a \cup b \in P$ implies $a \in P$ or $b \in P$ where as usual $a \cup b$ denotes the join of a and b.

Definition 1. *Two given elements of a semi-lattice are called U-separated, if there is an ultrafilter containing one element but not the other.*

Definition 2. *Two given elements of a lattice are called P-separated, if there exists a prime filter containing one element but not the other.*

In a semi-lattice L, every pair of distinct elements of L is U-separated if and only if L has disjunction property in the sense of Wallman.

For proof, see K. Iseki (1950, II).

By the notion of P-separability, we can characterize a distributive lattice:

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* For the definitions of lattices, Boolean algebra, see G. Birkhoff (1948).

A necessary and sufficient condition for a lattice L to be distributive is that every pair of distinct elements of L be P-separated.

For proof, see K. Iseki (1950, I or II).

The element a' is called the *complement of* a, if

(1) $a \cap a' = O$

(2) $a \cap x = O$ implies $x \leq a'$.

If every element of L has complement, we say that L is complemented semi-lattice. If U is an ultrafilter in such a semi-lattice, either $a \in U$ or $a' \in U$ for every a.

2. *Theorem of Stone-Samuel.*

Lemma 1. *If L is a complemented semi-lattice with disjunction property, $a'' = a$ for every element a in L, where $a'' = (a')'$.*

Proof. Obviously $a'' \geq a$. Suppose $a'' > a$, then there is an ultrafilter U such that $U \ni a''$ and $U \bar{\ni} a$. Since $U \bar{\ni} a$, $U \ni a'$ and $U \bar{\ni} a''$, which contradict. Therefore $a'' = a$.

Lemma 2. *A complemented semi-lattice such that $a'' = a$ is a lattice.*

Proof. We define $a \cup b = (a' \cap b')'$. We shall show that $a \cup b$ is the least upper bound of a and b. Since $a' \cap b' \leq a'$, $a \cup b = (a' \cap b')' \geq a'' = a$. Similarly $a \cup b \geq b$. Let x be a, $b \leq x$, then a', $b' \geq x'$. Hence $a \cup b = (a' \cap b')' \leq x'' = x$. This completes the proof. Therefore L is a lattice.

Lemma 3. *Let L be a lattice with disjunction property. If every ultrafilter in L is prime, L is distributive.*

Proof. Obvious from the theorem stated in §1.

Lemma 4. *A complemented semi-lattice L such that $a'' = a$ is a Boolean algebra with unit 1.*

Proof. We first show that L has disjunction property. Let a, b be any elements such that $a < b$. Let x be complement of a. If $b \cap x = O$, then $b \leq x' = a'' = a$, which contradicts $a < b$. Hence $b \cap x \neq O$ and clearly $a \cap x = O$. This shows L has disjunction property in the sense of Wallman. And by Lemma 2, L is a lattice.

Let U be an ultrafilter in L, if $a \cup b \in U$ and $a, b \bar{\in} U$, then $a', b' \in U$, hence $a' \cap b' \in U$, so we have $a \cup b = (a \cap b')' \bar{\in} U$. Therefore $a \cup b \in U$ implies $a \in U$ or $b \in U$. By the Lemma 3, L is distributive. $O' = 1$ is obvious. Hence by the well known theorem, L must be a Boolean algebra with unit 1.

Finally, we obtain the following theorem of Samuel (1948).

Theorem. *The following properties of a semi-lattice L are equivalent:*

(1) *L has disjunction property,*

(2) *$a'' = a$ for every element a in L.*

(3) *L is a Boolean algebra,*

(4) *every filter is the meet of all ultrafilters containing it.*

Proof. From the Lemma 1, (1) $\rightarrow$ (2), from the Lemma 4, (2) $\rightarrow$ (3), and since any Boolean algebra has disjuction property, (3) $\rightarrow$ (1). (4) $\rightarrow$ (1) is trivial. Hence it will suffice to prove (3) $\rightarrow$ (4). Let F be a filter. Suppose that there is an element a in

$L-F$ such that a is contained in every ultrafilter U containing F. $a' \bar{\epsilon} U \supset F$. Then there is an element x in F such that $a' \cap x = 0$. Hence $x \leq a'' = a$. Therefore $a \epsilon F$, which contradicts $a \bar{\epsilon} F$.

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BOOK REVIEW

A. Ostrowski: Vorlesungen uber Differential-und Integralrechnung, 2nd Vol. Differentialrechnung auf dem Gebiete mehrerer Variablen.
Verlag Birkhäuser, Basel 1951. Price: S. frcs. 63 (bound S. frcs. 67).

The book under review is the 2nd volume of a work on Calculus which is the result of Professor Ostrowski's profound knowledge and his long experience in teaching the subject. The 1st volume which was issued in 1945 (sold out at present) deals with the Calculus of functions of one variable; the 3rd volume is expected to be published soon and will contain the theory of integration of the functions of several variables. The subject matter of the 2nd volume is the differential calculus of functions of more than one variable. A great difficulty facing every writer of a text-book is the discrepancy between the systematical and the methodical approach to the subject. Systematically, one proceeds from the general to the particular, methodically, from the easy to the difficult. Perhaps there is no subject where this discrepancy is felt as much as in Calculus. A systematical representation may start from the theory of sets and—via topology, partially ordered sets and general theory of measure—may come to the analysis of the n-space and eventually to Calculus. Only a very few young students could benefit from such a course of lectures. On the other hand, a course on Calculus which as it is still prescribed in some curricula starts with the easiest portion, namely the technique of differentiation and which postpones the explanation of the meaning of those operations to a later date, is necessarily a curse to all the students with a mathematical brain The psychological tact of a born teacher and the profound knowledge of a scientist must be combined to produce a good book on Calculus. Moreover the right selection of the topics is a difficult task. Older textbooks attempted sometimes an encyclopaedean completeness which may fill the reader with awe and desperation. Modern writers of mathematical books are more inclined to err in the opposite direction; they use to give a well designed skeleton of their subject, but readers sometimes would prefer more "flesh" on it. It is very interesting to observe how the author of this work has attacked and solved his problem. He proceeds in steps starting with the easier portion of the subject (functions of one variable). Every chapter is treated in a systematical way and fully vigorous. There is no vagueness in the definition and no flaw in the conclusions. Nor is the reader startled by elegant tricks, but he is taken into confidence about the purpose of the investigation and the reason of the method applied, a way of proceeding which makes difficult things much easier. From numerous historical notes, the reader learns that mathematics is a living science. There are more than 1500 examples of all degrees of difficulty in the 2nd volume to be worked out by the student. This fact will certainly enhance the popularity of the book with the teachers and the examiners of mathematics in this country. The subject matter is restricted in a systematical way. Vol. I deals only with the notions: real numbers,

sequences of numbers and their convergence, continuous functions integration and derivatives. Such a limitation of the subject affords a through treatment and also a detailed representation of the technique of differentiation and integration. Complex numbers are not mentioned in Vol I and II, the subject being reserved for the course on function-theory. This restriction seems to be very wise; a superficial treatment of the complex with references to analogy with the real very often breeds loose reasoning. Moreover differential equations, calculus of variation and Fourier series are omitted since these topics are the subjects of other courses. The functions of bounded variation are discussed only in connection with the rectification of curves. They may perhaps be treated more explicitly in the 3rd volume The 2nd volume begins with sets and point-sets which are discussed as far as it is necessary for the later chapters. Schwarz' inequality, (Heine-)Borel's and Weierstrass' lemmas are proved. Ch. II deals with functions which are defined on a point-set of the n-space, especially their continuity. The 3rd chapter provides a very detailed discussion of the sequences and series of numbers and of functions. The power-series are singled out for a special treatment. The articles about the binomial series may be mentioned in particular; the chapter ends with the proof of Weierstrass' theorem about the approximation of continuous functions by polynomials. The Chapters IV and V contain the most important part of the book. The reviewer feels some envy with the students of the present generation because they are helped in this way to understand clearly such notions as "total differential", "Jacobian determinant" and "implicit function" and they get such a flawless and quite easily understandable representation of the solution of systems of equations. He still remembers how he had to struggle some forty years ago to bring some sense into the vague explanations he found in the books. The lucid representation of these topics in the book under review is supported by an able selection of notations; their translation into English has still to be done. Ch. VI reminds the reader that calculation is one of the roots of mathematics; it deals with numerical methods. Starting from linear interpolation and its application to interpolation into log-tables, the author proceeds to Lagrange's and Gregory-Newton's formulas (with residues). There after two methods of numerical differentiation (with an estimate of the error) are given. Numerical integration is discussed in a very detailed way (formula of the trapezoid, Simpson's rule, method of Gauss). The rule of Newton-Raphson for the approximative solution of equations is treated thoroughly. The last two chapters concern the application of Calculus to Geometry. Here, the author has deviated from his self-imposed rule to omit such topics as are discussed in other regular courses of lectures. He justifies this exception by the following remark in his preface: "To pass from a thoroughly elaborate course on Calculus to another on Differential Geometry, based on intuition, affords more mental strain for a student, than it may be expected". At the beginning of this century, the lectures on Geometry were in fact often lacking the necessary rigour. A similar laxity was also observed in some other lectures. Besides his personal experience (see above) the reviewer may quote E. Kamke who in the preface of his textbook on differential equations (1930) made the very outspoken remark that in quite a number of topics, the notions and proofs were lacking the rigour which other where in Analysis is

customary. It was just this experience which made him writing books on differential equations. The students of that time—the generation from which now-a-day's senior professors in Europe are recruited-felt mostly alike. Should one not expect them to give the benefit of their own experience to their students? "Intuition" (different from vagueness) has its proper place in teaching and research of Mathematics, otherwise there would not be any diagram in a good book on Calculus. The difference of the geometrical from the analytical point of view is not an illegitimate reference to spatial intuition, but the requirement of invariance for a group of transformations. This fact, every University-teacher of Geometry is expected to impress on his students. The reviewer was much inclined to charge the author with over-pessimism regarding the teaching of Geometry in Europe. when he came across a remark in the Mathematical Reviews (Vol. 12, p. 353) and learned that recently a famous Geometer has claimed a particular (apparently indefinable) kind of rigour ("substantial" instead of "formal" rigour) for Geometry. Therefore one must be afraid that the sad experience which made the author devote so much space to the applications to Geometry, does not date as far back as his student-days. At any rate those two chapters contain a host of useful information about that classical subject. Moreover they include some results of recent origin, in particular in the 500 examples offered to the reader for elaboration.

The book is written in a fluid style and the language will be easily understood by foreigners who have some knowledge of German. Printing and paper are excellent; a register (as was added to Vol. I) would make the book more comfortable for reference. Probably the register has been omitted for keeping the price down, but even now the book is beyond the financial means of most of the students and the badly underpaid staff of the Indian Universities. It is desirable that the teachers of Mathematics in this country will have the opportunity to borrow the volume from the libraries of the universities and other educational and scientific institutions since it is a particularly well-written and instructive book.

F. W. LEVI

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References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of the paper, the author's name following in alphabetical order, and each reference is to be arranged in the following form, *viz.*, name or names of authors ; year of publication ; name of the journal (abbreviation) ; number of volume ; and lastly, the page number. The following would be a useful illustration :

Wilson, B. M. (1922), *Proc. Lond. Math. Soc.* (2), 21, 235.

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